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## Generalized Ornstein-Uhlenbeck Processes in Ruin Theory

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# Thèse

en vue de l'obtention du grade de

## Docteur en Mathématiques appliquées

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Jérôme Spielmann


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# Abstract

This thesis is concerned with the study of Generalized Ornstein-Uhlenbeck (GOU) processes and their application in ruin theory. The GOU processes, which are the solutions of certain linear stochastic differential equations, have been introduced in ruin theory by Paulsen in 1993 as models for the surplus capital of insurance companies facing both insurance and market risks.

In general, these processes were chosen as suitable models on an *a priori* basis. The first and main contribution of this thesis is to show that GOU processes appear naturally as weak limits of random coefficient autoregressive processes which are used extensively in various domains of applied probability. Using this result, the convergence in distribution of the ruin times, the convergence of the ultimate ruin probability and the moments are also shown.

The rest of the thesis deals with the study of certain properties of GOU processes. In particular, the ruin problem for the GOU process is studied and new bounds on the ruin probabilities are obtained. These results also generalize some known upper bounds, asymptotic results and conditions for certain ruin to the case when the market risk is modelled by a semimartingale.

The final section of the thesis moves away from classical ruin theory and lays some first directions for the study of the law of GOU processes at fixed times. In particular, a partial integro-differential equation for the density, large and small-time asymptotics are obtained for these laws. This shift away from the ruin probability is explained by the fact that most risk measures used in practice such as Value-at-Risk are based on these laws instead.

# Resumé

Cette thèse contribue à l'étude des processus d'Ornstein-Uhlenbeck généralisés (GOU) et de leurs applications en théorie de la ruine. Les processus GOU, qui sont les solutions de certaines équations différentielles stochastiques linéaires, ont été introduits en théorie de la ruine par Paulsen en 1993 en tant que modèles pour le capital d'une assurance soumise au risque de marché.

En général, ces processus ont été choisis comme modèles de manière *a priori*. La première et principale contribution de cette thèse est de montrer que les processus GOU apparaissent de manière naturelle comme limites faibles de processus autoregressifs à coefficients aléatoires, processus qui sont très utilisés en probabilité appliquée. À partir de ce résultat, la convergence en distribution des temps de ruine, la convergence des probabilités de ruine ainsi que la convergence des moments sont aussi démontrées.

Le reste de la thèse traite de certaines propriétés des processus GOU. En particulier, le problème de la ruine est traité et de nouvelles bornes sur les probabilités de ruine sont obtenues. Ces résultats généralisent aussi des résultats connus au cas où le risque de marché est modélisé par une semi-martingale.

La dernière partie de la thèse s'éloigne de la théorie de la ruine pour passer à l'étude de la loi du processus à temps fixe. En particulier, une équation intégrodifférentielle partielle pour la densité est obtenue, ainsi que des approximations pour la loi en temps courts et longs. Cet éloignement s'explique par le fait que la plupart des mesures de risque utilisées dans la pratique sont basées sur ces lois et non sur la probabilité de ruine.

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# Contents

# Introduction

Ruin theory is the part of actuarial science that studies the possibility of insolvency associated with insurance practice. It is one of the oldest domains of actuarial science and, in fact, its history traces back to the Enlightenment and the development of probability theory, see e.g. [Pradier, 2003]. The main questions which drive this research are: what is a good mathematical model for the capital or surplus of an insurance company and, given some good model, what is the probability of ruin or insolvency ?

We start by looking at the first of these questions. The part of ruin theory that interests us specifically concerns the use of *(continuous-time) stochastic processes*.[1] While this restriction gives a direction, the collection of continuous-time stochastic processes is still very large and further considerations are needed to obtain a candidate model. A standard approach is to consider the risks associated with insurance practice and to choose a model which is both relatively realistic and tractable.

In the context of financial risks, multiple different causes exist which differentiate the types of risks faced by insurance companies. The Basel Committee on Banking Supervision[2], for example, distinguishes *insurance or underwriting risk* due to the randomness in the times and sizes of payments to the policy-holders, *operational risk* due to malfunction of business processes, *market risk* due to random fluctuations of asset values, *credit risk* due to the possibility of insolvency of a counterpart, etc... The question of the choice of

[1] See Section 1.1 for the definition of stochastic processes. In this thesis, we will focus on *continuous-time* processes since, in addition to being in agreement with our experience of time, the use of such processes allows the use of the powerful Itô or stochastic calculus to study their properties.

[2] See e.g [Basel Committee on Banking Supervision, 2004] for a more precise definitions of some of these risks.

a particular stochastic process is thus related to the question of what types of risks to take into account.

In the seminal thesis [Lundberg, 1903], it is assumed that the only risk faced by a company is due to underwriting insurance and the model of insurance capital is given by the *compound Poisson process with drift*

$$X_t = y + pt - \sum_{i=1}^{N_t} Z_i, \ \ t \geq 0, \tag{1}$$

where $y > 0$ represents the initial capital, $p > 0$ is a constant representing the mean income received from payments by the policy-holders in some unit of time, $N = (N_t)_{t \geq 0}$ is a standard Poisson process representing the random number and times of payments to the policy-holders in case of damage and $(Z_i)_{i \in \mathbb{N}^*}$ is a sequence of non-negative random variables representing the random sizes of these payments. The study of this model, which is usually called the *Cramér-Lundberg model*, and its generalizations has given rise to a vast body of literature which is thoroughly reviewed in [Asmussen and Albrecher, 2010].

Increasingly, the recognition that insurance companies invest their capital in financial assets has lead to the addition of market risk to the model. Then, two stochastic processes $X = (X_t)_{t \geq 0}$ and $R = (R_t)_{t \geq 0}$ are considered and the model is defined as the solution of the following stochastic differential equation (see Section 1.2.2 for the definition of these equations)

$$Y_t = y + X_t + \int_{0+}^{t} Y_{s-} dR_s, \ \ t \geq 0, \tag{2}$$

where $y > 0$ represents the initial capital, $X$ represents the profit and loss due to underwriting insurance and the integral represents the profit and loss of investment at rate $R$ up to time $t$.[3] Up to our knowledge, the use of such general models, in the context of ruin theory, was suggested for the first time in [Paulsen, 1993]. The solution of Equation (2) is, under some conditions (see Section 1.3), given by the *generalized Ornstein-Uhlenbeck (GOU)*

[3] The form of equation (2) suggests that at each time the entirety of the capital of the company is invested in a financial asset, but this depends simply on the definition of $R$. For example, if the company only invests a fraction $\alpha \in (0, 1)$ of its capital we could use the process $(\alpha R_t)_{t \geq 0}$ in (2) instead of $R$.

*process*

$$Y_t = \mathcal{E}(R)_t \left( y + \int_{0+}^{t} \mathcal{E}(R)_{s-}^{-1} dX_s \right), \ \ t \geq 0, \tag{3}$$

where $\mathcal{E}(.)$ is the stochastic or Doléans-Dade exponential (see Proposition 1.2.10 for the definition).

The main goal of this thesis is to motivate GOU processes as candidate models, which thus incorporate both insurance and market risks, and to contribute to the study of some of their properties. This is done in three parts (Chapters 2, 3, and 4) whose contents we now describe.

In actuarial mathematics, the choice of Equation (2) or of the GOU process (3) as a model was mostly based on an *a priori* choice. The main goal of Chapter 2 is to show that it is indeed a model that appears naturally as a limit of discrete-time processes which are used extensively in applied probability and in ruin theory, in particular. More precisely, consider the *random coefficient autoregressive process of order one* which is denoted RCA(1) and defined recursively by $\theta_0 = 0$ and

$$\theta_k = \xi_k + \theta_{k-1}\rho_k, \ \ k \in \mathbb{N}^*,$$

where $(\xi_k)_{k\in\mathbb{N}^*}$ and $(\rho_k)_{k\in\mathbb{N}^*}$ are two sequences of random variables. We prove that, under additional assumptions, a certain (continuous-time) renormalization $\theta^{(n)} = (\theta_t^{(n)})_{t\geq 0}$, with $n \in \mathbb{N}^*$, of this process converges in distribution to a GOU process of the form (3) when the time-step goes to 0 and where $X$ and $R$ are stable Lévy processes (see Theorem 2.2.1 for the result and the beginning of Section 2.2 for the details). In mathematics, such convergence theorems are known as *invariance principles*.

To further describe the relevance of this result to ruin theory, we now introduce the *ruin time* which is the stopping time (see Section 1.1 for the definition of stopping times) that corresponds to the first time a stochastic process goes below 0. More precisely, the ruin times of $\theta^{(n)}$ are defined as

$$\tau^n(y) = \inf\{t > 0 : \theta_t^{(n)} < 0\}, \ \ n \in \mathbb{N}^*,$$

and the ruin time of the GOU process as

$$\tau(y) = \inf\{t > 0 : Y_t < 0\},$$

where $y > 0$ represents the initial capital and with the convention that $\inf\{\emptyset\} = \infty$. Equipped with these definitions, we continue in Chapter 2 by applying our previous invariance principle to prove the convergence in distribution of the ruin times $\tau^n(y)$ to $\tau(y)$, as $n \to \infty$ (Theorem 2.2.3). Equivalently, this proves that, for all $T > 0$,

$$\lim_{n\to\infty} \mathbf{P}(\tau^n(y) \leq T) = \mathbf{P}(\tau(y) \leq T).$$

The *(finite-time) ruin probability* $\mathbf{P}(\tau(y) \leq T)$, which represents the probability of ruin before some time $T > 0$, is fundamental to ruin theory. An other quantity whose estimation is of importance is the *ultimate or infinite-time ruin probability* $\mathbf{P}(\tau(y) < \infty)$ which can be interpreted as the probability that insolvency happens (without specifying a time-frame). In general, the results for these quantities depend on $y$ and thus give a theoretical control on the possibility of ruin via the initial capital. We finish Chapter 2 by proving that the ultimate ruin probabilities (Theorem 2.3.1) and the moments (Theorem 2.4.2) of $\theta^{(n)}$ also converge to those of $Y$, in the case when $\xi_1$ and $\ln(\rho_1)$ are square-integrable. In this simpler case, we also give the explicit values of the limiting quantities and thus give a way to approximate them for RCA(1) processes when the steps between updates and their magnitudes are small.

Before describing the results of the following chapters, we turn back for a moment to the Cramér-Lundberg model (1). The most well known result about this model is that the ultimate ruin probability decreases at least exponentially fast when $y$ increases, when the *safety loading condition* $\mathbf{E}(X_1) > 0$ is satisfied, and is equal to 1, for all $y > 0$, when this condition is not satisfied. This naturally leads to the question of how the ultimate ruin probability behaves in the context of more general GOU models.

The surprising discovery in [Frolova et al., 2002], [Kalashnikov and Norberg, 2002] and [Paulsen, 2002] is that, in this case, the ultimate ruin probability decreases as a power function when $y$ increases and that the safety loading condition is replaced by a condition on $R$ which separates the states of the world into a *large volatility* and *small volatility* state. These results thus show that the decrease of the ultimate ruin probability is slower when market risk is added to the model, which is in agreement with our intuition. However, the shift from the safety loading condition to the volatility condition is surprising. This means, paradoxically, that the possibility of ruin depends on the state of the market (and the investment strategy) and not on the way the insurance business is handled ; if we are in a small volatility state, the ultimate ruin probability will decrease to 0 as $y$ increases, even if insurance policies are sold at a loss and, conversely, ruin will be certain, for all $y > 0$, in a large volatility state, even if the insurance business is managed perfectly. Since these seminal papers appeared, the ruin problem for GOU processes has been extensively studied (see 3.2 for a summary of the known results). Here, we simply mention that these results are in general concerned with the case where $X$ and $R$ are Lévy processes (see Definition 1.2.4 for the definition of these processes).

In Chapter 3, we thus look at some extensions of these results. In view of the previous discussion, we see that the focus for ruin theory, in this context, should be on the returns process and the investment strategy of the insurance company. In mathematical finance, the returns of an investment strategy are modelled by a stochastic integral which is a semimartingale (see Definition 1.2.1 for the definition of this object). Thus, the first contribution of Chapter 3 is to extend some results which were known when $R$ is Lévy process to the more general case when $R$ is a semimartingale (see Corollary 3.2.2 for the generalization of an upper bound on the ultimate ruin probability and Theorem 3.5.1 for the generalization of a result on conditions for certain ruin). This generalization also allows to consider more realistic markets which change over time or switch between different states.

The second contribution of Chapter 3 lies in the focus on the *finite-time* ruin probability. Indeed, most of the literature on this subject considers the ultimate ruin probability instead. We believe that the finite-time ruin probability is much more important than the ultimate ruin probability, since it is more important to know if an insurance company will go bankrupt before the age of retirement of some generation, rather than to know if it will go bankrupt before the explosion of the sun. Thus, we obtain an upper bound for the finite-time ruin probability (Theorem 3.2.1) and prove that it is asymptotically optimal in some sense and in a large number of cases (Theorem 3.3.1).

While the ruin probability is very important both from a theoretical and practical point of view, most risk measures used in practice are based on the law of $Y_t$, for some fixed time $t > 0$. For example, value-at-risk and other quantile based risk measures play a central role in the Basel regulatory framework, see e.g. [Basel Committee on Banking Supervision, 2019]. Thus we give, in Chapter 4, some directions for the study of the law of $Y_t$, for fixed $t > 0$. In particular, we use the theory of Markov processes to obtain a partial integro-differential equation for the density when it exists (Theorem 4.2.3) and give sufficient conditions for its existence (Proposition 4.2.4). Since this equation seems hard to solve analytically even in simple cases, we then study approximations of the law of $Y_t$ when $t$ is either small or large. This leads, in particular, to the identification of a normal-log-normal mean-variance mixture as a suitable small-time approximation (Section 4.3) and a log-normal distribution (in the small volatility case) as a suitable large-time approximation for the positive (and negative) parts of the law of $Y_t$ (Section 4.4).

# Chapter 1

# Mathematical Definitions and Basic Facts

In this chapter, we introduce the notations, mathematical objects and basic facts that will be used throughout the thesis. The material in Sections 1.1 and 1.2 is standard and we will, in general, follow [Jacod and Shiryaev, 2003] for the presentation. The material in Section 1.3 is less standard and we will give the details for completeness.

## 1.1 Preliminaries and Notations

We are given a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ which we can equip with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, so that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ forms a *stochastic basis*. Unless stated otherwise, we will assume that the filtration $\mathbb{F}$ is complete (i.e. $\mathcal{F}_0$ contains all $\mathbf{P}$-null sets of $\mathcal{F}$) and right-continuous (i.e. $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$, for all $t \geq 0$).

A *stochastic process* is a collection $S = (S_t)_{t\geq 0}$ of mappings $S_t : \Omega \to \mathbb{R}$.[1] We can also consider stochastic processes as mappings of $\Omega \times \mathbb{R}_+$ into $\mathbb{R}$ via

[1]Since in this thesis we are interested in modelling a single insurance company and not effects between them, we present the facts for stochastic processes valued in $\mathbb{R}$ rather than $\mathbb{R}^d$ or some more general space.

the map $(\omega, t) \mapsto S_t(\omega)$, for each $(\omega, t) \in \Omega \times \mathbb{R}_+$. For a fixed $\omega \in \Omega$, the mapping $t \mapsto S_t(\omega)$ is called a *path of* $S$.

A stochastic process is *measurable* (respectively *progressively measurable*) if the mapping $(\omega, t) \mapsto X_t(\omega)$ is $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$-measurable (respectively $\mathcal{F}_t \otimes \mathcal{B}([0,t])$-measurable, for all $t \geq 0$), where $\mathcal{B}(E)$ is the Borel $\sigma$-algebra associated to some topological space $E$. We say that $S$ is *adapted* to $\mathbb{F}$, if, for each $t \geq 0$, the mapping $\omega \mapsto S_t(\omega)$ is $\mathcal{F}_t$-measurable. The *natural filtration* $\mathbb{F}^S = (\mathcal{F}^S)_{t\geq 0}$ of $S$, given by $\mathcal{F}_t^S = \sigma(S_s, 0 \leq s \leq t)$, is the smallest filtration to which $S$ is adapted.[2]

We say that a stochastic processes is *càdlàg* if there exists a set $\Omega_0$ of probability one, such that the path $t \to S_t(\omega)$ is right-continuous with left-hand limits, for all $\omega \in \Omega_0$. Similarly, we say that a stochastic process is *continuous* (respectively *left-continuous*) if the paths of the process are continuous (respectively left-continuous) on a set of probability one. Given a càdlàg stochastic process $S = (S_t)_{t\geq 0}$, we define the two following processes $S_- = (S_{t-})_{t\geq 0}$ as $S_{0-} = S_0$ and $S_{t-} = \lim_{s\nearrow t} S_s$ and the *jumps* $\Delta S = (\Delta S_t)_{t\geq 0}$ as $\Delta S_t = S_t - S_{t-}$.

The space of adapted càdlàg functions $\mathbb{R}_+ \to \mathbb{R}$ is denoted $\mathbb{D}$, the space of adapted left-continuous functions with right-hand limits $\mathbb{L}$ and the space of adapted continuous functions $\mathcal{C}$. The notation $\mathcal{C}^k$ denotes the space of $k$-times continuously differentiable functions and $\mathcal{C}^{k,l}$ the space of functions with domain $\mathbb{R}_+ \times \mathbb{R}_+$ which are $k$-times differentiable in the first variable and $l$-times in the second. The notation $\mathcal{C}_0^k$ denotes the space of $k$-times differentiable functions which vanish at infinity. When the domain is not $\mathbb{R}_+$ (or $\mathbb{R}_+ \times \mathbb{R}_+$) but some other space $B$, we write $\mathcal{C}(B)$ to designate the space of continuous function with domain $B$ with the same convention for the other functional spaces.

A *stopping time* $\tau$ is a random variable $\tau : \Omega \to [0, \infty]$ such that $\{\tau \leq t\} \in \mathcal{F}_t$, for all $t \geq 0$.[3]

[2] Note that this filtration is not necessarily complete and right continuous. When we extend it to satisfy these assumptions, it is in general called the *augmented natural filtration*. However, in this thesis we will always take the augmented natural filtration and we call it simply the *natural filtration*.

[3] Note that we allow stopping times to take an infinite value. We will use the term *finite stopping time* when the stopping time is almost surely finite.

We will write $\stackrel{a.s.}{\to}$ for the almost sure convergence, $\stackrel{\mathbf{P}}{\to}$ for the convergence in probability, $\stackrel{d}{\to}$ for the convergence in law and $\stackrel{d}{=}$ for the equality in law. We write $\mathbf{E}(Z)$ for the expectation of some random variable $Z$ and $\mathrm{Var}(Z)$ for its variance. A generic normal random variable with expectation $\mu$ and variance $\sigma^2$ is denoted $\mathcal{N}(\mu, \sigma^2)$. The notation $\mathcal{L}(Z)$ designates the law of the random variable $Z$ and $\mathbf{E}(Z|\mathcal{G})$ denotes the conditional expectation of $Z$ with respect to some event, random variable or $\sigma$-algebra $\mathcal{G}$. When $F$ is some set, we denote by $\mathbf{1}_F$ the indicator function of $F$. Finally, for any $x, y \in \mathbb{R}$, $x \wedge y$ means the minimum of $x$ and $y$, $x \vee y$ the maximum, $(x)^+$ means $x \vee 0$, and $x \mapsto [x]$ is the floor function.

**Abbreviations :** *i.i.d.* stands for "independent and identically distributed", ($\mathbf{P}-a.s.$) stands for almost surely for the probability measure $\mathbf{P}$, *w.r.t.* stands for "with respect to" and *r.h.s.* and *l.h.s.* stand for "right-hand side" and "left-hand side". The abbreviation "SDE" stands for *stochastic differential equation* as defined in Section 1.2.2. A *PII* is a "process with independent increments" (see Definition 1.2.4) and *GOU process* stands for "generalized Ornstein-Uhlenbeck process" (see Section 1.3).

## 1.2 Semimartingales and Stochastic Calculus

Intuitively, a semimartingale is composed of two parts: a finite variation process and a local martingale. In this section, after defining these terms, we introduce the notions of quadratic variation and characteristics of semimartingales and present the fundamental example of the Lévy processes.

### 1.2.1 Martingales, Finite Variation Processes and Semimartingales

First, a martingale $M = (M_t)_{t\geq 0}$ (respectively a submartingale, a supermartingale) is a càdlàg stochastic process, which is adapted to some filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and satisfies $\mathbf{E}(|M_t|) < \infty$, for each $t \geq 0$, and $\mathbf{E}(M_t|\mathcal{F}_s) = M_s$ (respectively $\mathbf{E}(M_t|\mathcal{F}_s) \geq M_s$, $\mathbf{E}(M_t|\mathcal{F}_s) \leq M_s$) ($\mathbf{P} - a.s.$), for each $s \leq t$. A martingale $M$ is *uniformly integrable* if

$$\sup_{t\geq 0} \mathbf{E}(|M_t|\mathbf{1}_{\{|M_t|\geq n\}}) \to 0, \text{ as } n \to \infty, \tag{1.1}$$

and it is *square-integrable* if $\sup_{t\geq 0} \mathbf{E}(|M_t|^2) < \infty$. It is possible to check that a square-integrable martingale is also uniformly integrable using de la Vallée-Poussin's criterion.[4]

A *local martingale* (respectively a *locally square integrable martingale*) is a càdlàg stochastic process, which is adapted to some filtration and for which there exists a non-decreasing sequence of finite stopping times $(\tau_n)_{n\in\mathbb{N}}$ with $\tau_n \nearrow \infty$, as $n \to \infty$, such that the *stopped process* $M^{\tau_n} = (M_{t\wedge\tau_n}\mathbf{1}_{\{\tau_n>0\}})_{t\geq 0}$ is a uniformly integrable martingale (respectively a square-integrable martingale), for each $n \in \mathbb{N}$. Such a sequence of stopping times is called a *localizing sequence.*

Given a càdlàg and adapted stochastic process $A = (A_t)_{t\geq 0}$ on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$, we can define its *first order variation over* $[0, T]$ (also called *total variation*), for each $\omega \in \Omega$, as

$$V(A)_T(\omega) = \sup \sum_{i=0}^{n-1} |A_{t_i+1}(\omega) - A_{t_i}(\omega)|,$$

where the supremum is taken over all partitions $0 \leq t_0 < t_1 < \cdots < t_n = T$ of $[0, T]$. Then, we say that $A$ has *finite variation over each finite interval* (or simply *finite variation*) if $\sup_{T\geq 0} V(A)_T(\omega) < \infty$, for all $\omega \in \Omega$, and that $A$ has *integrable variation* if $\mathbf{E}(V(A)_\infty) < \infty$. Using a localizing sequence $(\tau_n)_{n\in\mathbb{N}}$, we can define the notion of *locally integrable variation* by asking that $V(A)^{\tau_n}$ has integrable variation, for each $n \in \mathbb{N}$.[5]

We are now ready to give the definition of a semimartingale which will be used throughout the thesis.

[4] de la Vallée-Poussin's criterion : if there exists a measurable function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{x\to\infty} \phi(x)/x = \infty$ and $\sup_{t\geq 0} \mathbf{E}(\phi(|M_t|)) < \infty$, then (1.1) is satisfied.

[5] The notion of *locally finite variation* is not needed, since it is equivalent to the notion of *finte variation.* Note also that some authors use the terminology *bounded variation* instead of *finite variation.*

**Definition 1.2.1** (Semimartingales and special semimartingales)**.** *A semimartingale $S = (S_t)_{t\geq 0}$ on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ is a càdlàg stochastic process of the form*

$$S_t = X_0 + A_t + M_t, \ \ t \geq 0, \tag{1.2}$$

*where $M = (M_t)_{t\geq 0}$ is a local martingale with $M_0 = 0$ and $A = (A_t)_{t\geq 0}$ is a process with finite variation with $A_0 = 0$, both adapted w.r.t. the filtration $\mathbb{F}$. Moreover, a semimartingale $S$ is special if $A$ is a process with locally integrable variation.*

In some texts, the condition for $S$ to be a special semimartingale is replaced by the requirement that $A$ is *predictable*, i.e. measurable w.r.t. the $\sigma$-field $\mathcal{P}$ on $\Omega \times \mathbb{R}_+$ generated by the left-continuous adapted processes.[6] In fact, these definitions can be shown to be equivalent (see e.g. Proposition 4.23 p.44 in [Jacod and Shiryaev, 2003]). The predictability of $A$ guarantees the uniqueness (up to indistinguishability[7]) of the decomposition (1.2), which is then called *the canonical decomposition* of $S$.

### 1.2.2 Stochastic Integral and Quadratic Variation

To describe stochastic processes we need, in general, to go further than first order variation. In order to define this notion of second order variation, we now briefly recall the definition of the stochastic or Itô integral and the notion of stochastic differential equations.

We start by considering the class $\mathfrak{S}$ of *simple predictable* stochastic processes of the form

$$H_t = H_0 \mathbf{1}_{\{0\}} + \sum_{i=1}^{n} H_i \mathbf{1}_{\{(t,.)\in ]\!]\tau_i, \tau_{i+1}]\!]\}}, \ \ t \geq 0$$

where $0 = \tau_1 \leq \tau_2 \leq \cdots \leq \tau_{n+1} < \infty$ ($\mathbf{P}-a.s.$) is a sequence of stopping times such that, for each $i = 1, \ldots, n+1$, $H_{\tau_i}$ is $\mathcal{F}_{\tau_i}$-measurable[8] and $|H_i| < \infty$

[6] In the rest, we will also need the notion of *optionnality* (i.e. $A$ is *optional*) which means measurable w.r.t. the $\sigma$-field $\mathcal{O}$ on $\Omega \times \mathbb{R}_+$ generated by the càdlàg adapted processes.

[7] Two stochastic processes $X$ and $Y$ are *indistinguishable* if $\mathbf{P}(X_t = Y_t, \forall t \geq 0) = 1$.

[8] Given a stopping time $\tau$ for some filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, we can define the *stopped $\sigma$-algebra* $\mathcal{F}_\tau$ as $\mathcal{F}_\tau = \{A \in \mathcal{F} | A \cap \{\tau \leq t\} \in \mathcal{F}_t, t \geq 0\}$.

$(\mathbf{P} - a.s.)$.[9] For any càdlàg process $S = (S_t)_{t\geq 0}$, we can then define the application $I_S : \mathfrak{S} \to \mathbb{D}$ as

$$I_S(H)_t = H_0 S_0 + \sum_{i=1}^{n} H_i (S_t^{\tau_{i+1}} - S_t^{\tau_i}),\ \ t \geq 0.$$

It is possible to show that $\mathfrak{S}$ is dense in $\mathbb{L}$ for the so-called ucp topology (see p.57 in [Protter, 2005] for the definition of this topology and for the density result) and that, when $S$ is a semimartingale[10] the application $I_S$ is continuous for this topology (see Theorem 11 p.58 in [Protter, 2005]). Thus, it is possible to extend uniquely (up to indistinguishability) $I_S$ as linear mapping on $\mathbb{L}$, i.e. $I_S : \mathbb{L} \to \mathbb{D}$, which we call the *stochastic integral of $H$ w.r.t. $S$* and which we denote $I_S(H) = \int_0^{\cdot} H_s dS_s$ and $I_S(H) = \int_{0+}^{\cdot} H_s dS_s$, when we want to exclude 0. We refer to Section II.4 in [Protter, 2005] for more informations and details about the stochastic integral.

A stochastic differential equation (SDE) is an equation relating the value of a process with the value of the stochastic integral of this process. More precisely, if $S = (S_t)_{t\geq 0}$ is some semimartingale with $S_0 = 0$ and $f : \Omega \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ some function, we can write the equation

$$Y_t = Y_0 + \int_{0+}^{t} f(., s, Y_{s-}) dS_s,\ \ t \geq 0,$$

where $Y_0$ is some $\mathcal{F}_0$-measurable random variable, which we call a *stochastic differential equation (SDE)*. We will also often denote this expression by

$$dY_t = f(., t, Y_{t-}) dS_t,\ \ t \geq 0$$

with $Y_0 = Y_0$ $(\mathbf{P} - a.s.)$. For the general results on the existence and uniqueness of the solutions of these equations see e.g. Section V.3 starting p.255 in [Protter, 2005].

[9] Here $\rrbracket \tau_i, \tau_{i+1} \rrbracket$ is a stochastic interval. A *stochastic interval* is a subset of $\mathbb{R}_+ \times \Omega$ of the form $\llbracket \tau_1, \tau_2 \rrbracket = \{(t, \omega) : \tau_1(\omega) \leq t \leq \tau_2(\omega)\}$, where $\tau_1$ and $\tau_2$ are two random times. The preceding expression defines the *closed* stochastic interval. The open $\rrbracket \tau_1, \tau_2 \llbracket$ and half-open intervals $\llbracket \tau_1, \tau_2 \llbracket$ and $\rrbracket \tau_1, \tau_2 \rrbracket$ are defined in a similar way.

[10] Note that Protter's definition of *semimartingale* is different from ours. Definition 1.2.1 corresponds to what they call *classical semimartingale* and the two definitions are, in fact, equivalent (see Theorem 47 p.146 in [Protter, 2005]).

The *quadratic covariation* (or *square bracket*) of two semimartingales $S$ and $U$ is then given by

$$[S,U]_t = S_t U_t - S_0 U_0 - \int_{0+}^{t} U_{s-} dS_s - \int_{0+}^{t} S_{s-} dU_s, \ \ t \geq 0 \tag{1.3}$$

and $[S,S]$ is called the *quadratic variation* of $S$ which we write $[S,S]_t = [S]_t$, for all $t \geq 0$.

It is possible to obtain a simpler expression for the square-bracket as a sum of a predictable part and of a jump process. For this recall that given two locally square-integrable martingales $M$ and $N$, it is possible to define their *predictable quadratic covariation* (or *angle bracket*) as the unique (up to indistinguishability) predictable process $\langle M,N\rangle = (\langle M,N\rangle_t)_{t\geq 0}$ with locally integrable variation such that $(M_t N_t - \langle M,N\rangle_t)_{t\geq 0}$ is a local martingale. The process $\langle M,M\rangle$ is called the *predictable quadratic variation* of $M$ and we write $\langle M\rangle_t = \langle M,M\rangle_t$, for all $t \geq 0$. Note that a continuous local martingale is also locally square-integrable and, thus, that the angle bracket is defined also when $M$ and $N$ are continuous local martingales.

It is known that any local martingale $M$ can be decomposed uniquely (again up to indistinguishability) as $M_t = M_0 + M_t^c + M_t^d$, $t \geq 0$, where $M_0$ is an $\mathcal{F}_0$-measurable random variable, $M^c = (M_t^c)_{t\geq 0}$ is a continuous local martingale and $M^d = (M_t^d)_{t\geq 0}$ is a purely discontinuous local martingale[11] (see e.g. Theorem 4.18 p.42-43 in [Jacod and Shiryaev, 2003]). The process $M^c$ is then called the *continuous part* of $M$. Thus, to any semimartingale $S$ we can associate its *continuous martingale part* $S^c$ which corresponds to the continuous martingale part of the local martingale in any decomposition of the form (1.2).[12] Using these additional facts, it is possible to show that

$$[S,U]_t = \langle S^c, U^c\rangle_t + \sum_{0<s\leq t} \Delta S_s \Delta U_s, \ \ t \geq 0, \tag{1.4}$$

(see e.g. Theorem 4.52 p.55 in [Jacod and Shiryaev, 2003]).

[11] A local martingale $M = (M_t)_{t\geq 0}$ is *purely discontinuous* if $M_0 = 0$ and $(M_t N_t)_{t\geq 0}$ is a local martingale for any continuous local martingale $N = (N_t)_{t\geq 0}$.

[12] To prove that it is unique (up to indistinguishability) take two possible decompositions $S_t = S_0 + A_t + M_t$ and $S_t = S_0 + \tilde{A}_t + \tilde{M}_t$, for all $t \geq 0$. Let $N_t = A_t - \tilde{A}_t = \tilde{M}_t - M_t$, $t \geq 0$. Then, by definition $N$ is a local martingale and a finite variation process, and, by Lemma 4.14 p.41 in [Jacod and Shiryaev, 2003], it is thus purely discontinuous. So, $M_t^c = \tilde{M}_t^c$, for all $t \geq 0$.

We can now state the famous Itô formula for semimartingales (see Theorem 4.57 p.57 in [Jacod and Shiryaev, 2003] for the proof).

**Lemma 1.2.2** (Itô's formula). *Let $S = (S_t)_{t\geq 0}$ be a semimartingale and $f \in \mathcal{C}^2(\mathbb{R})$. Then, $(f(S_t))_{t\geq 0}$ is a semimartingale and*

$$\begin{aligned} f(S_t) = f(S_0) &+ \int_{0+}^{t} f'(S_{s-})dS_s + \frac{1}{2}\int_{0+}^{t} f''(S_{s-})d\langle S^c\rangle_s \\ &+ \sum_{0<s\leq t} \left(f(S_s) - f(S_{s-}) - f'(S_{s-})\Delta S_s\right), \quad t \geq 0. \end{aligned}$$

### 1.2.3 Random Measures, Compensators and Compensated Integrals

In this section, we introduce the notions of random measures and their compensators. These will be used to describe the jumps of a stochastic process by encoding the random numbers and sizes of jumps.

First, a *random measure* on $\mathbb{R}_+ \times E$[13] is a family $\mu = (\mu(\omega; dt, dx))_{\omega\in\Omega}$ of measures on $(\mathbb{R}_+ \times E, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(E))$ (where the $\sigma$-algebra is the tensor product of the Borel $\sigma$-algebras) satisfying $\mu(\omega; \{0\} \times E) = 0$, for all $\omega \in \Omega$. Recall that $\mathcal{P}$, respectively $\mathcal{O}$, denote the predictable, respectively optional, $\sigma$-algebras on $\Omega \times \mathbb{R}_+$. Since a random measure is a measure for each $\omega \in \Omega$, we can define its *integral process* $H * \mu = ((H * \mu)_t)_{t\geq 0}$, for any function $H$ on $\Omega \times \mathbb{R}_+ \times E$ which is $\mathcal{O} \otimes \mathcal{B}(E)$-measurable, by

$$H * \mu_t(\omega) = \int_0^t \int_E H(\omega, s, x)\mu(\omega; ds, dx), \tag{1.5}$$

if $\int_0^t \int_E |H(\omega, s, x)|\mu(\omega; ds, dx) < \infty$ and by $H * \mu_t(\omega) = \infty$ otherwise, for all $t \geq 0$.

Rather than working directly with random measures, it is often useful to consider their *predictable compensator* (or simply *compensator*) which are

[13] Here the space $(E, \mathcal{B}(E))$, where $\mathcal{B}(E)$ is is the Borel $\sigma$-algebra on $E$, has to be a *Blackwell space*, see Chapitre III in [Dellacherie and Meyer, 1975] for a definition. In the rest of the thesis, $(E, \mathcal{B}(E))$ will be $\mathbb{R}$ with its Borel $\sigma$-algebra which is a Blackwell space and represents the space of the jump sizes of some stochastic process.

defined in the following proposition (for the proof see Theorem 1.8 p.66-67 in [Jacod and Shiryaev, 2003]).

**Proposition 1.2.3** (Compensation formula)**.** *Given a random measure $\mu$ satisfying some additional conditions*[14]*, there exists a random measure, called the compensator of $\mu$ and denoted $\mu^p$, which is unique up to a $\mathbf{P}$-null set and is defined by the following equivalent conditions :*

1. $\mathbf{E}(H * \mu_\infty) = \mathbf{E}(H * \mu^p_\infty)$*, for any function $H$ on $\Omega \times \mathbb{R}_+ \times E$ which is $\mathcal{P} \otimes \mathcal{B}(E)$-measurable,*
2. *for any function $H$ on $\Omega \times \mathbb{R}_+ \times E$ which is $\mathcal{P} \otimes \mathcal{B}(E)$-measurable and such that $|H| * \mu$ is a (càdlàg adapted non-decreasing) locally integrable processes, $|H| * \mu^p$ is a locally integrable processes and $H * \mu - H * \mu^p$ is a local martingale.*

*Moreover, there exists a predictable integrable non-decreasing process $A$ and a kernel $K(\omega; t, dx)$ from $(\Omega \times \mathbb{R}_+, \mathcal{P})$ to $(E, \mathcal{B}(E))$ such that*

$$\mu^p(\omega; dt, dx) = dA_t(\omega) K(\omega; t, dx). \tag{1.6}$$

One of the most useful examples of a random measure is the *jump measure* associated with a càdlàg adapted stochastic process $S$ which is defined on $\mathbb{R}_+ \times \mathbb{R}$ by

$$\mu_S(\omega; dt, dx) = \sum_{s>0} \mathbf{1}_{\{\Delta S_s(\omega) \neq 0\}} \delta_{(s, \Delta S_s(\omega))}(ds, dx),$$

where $\delta$ is the Dirac measure at some point in $\mathbb{R}_+ \times \mathbb{R}$. Intuitively, this measure counts the number of jumps of different sizes in some subset of time. We have, given any function $H$ for which (1.5) is defined, that

$$\int_{0+}^t \int_{\mathbb{R}} H(\omega, s, x) \mu_S(\omega; ds, dx) = \sum_{0<s\leq t} \mathbf{1}_{\{\Delta S_s(\omega) \neq 0\}} H(\omega; s, \Delta S_s(\omega)), \quad t \geq 0.$$

In fact, for the jump measure $\mu_S$ there exists a version of the compensator (see Proposition 1.17 p.70 in [Jacod and Shiryaev, 2003]) with some nice

[14] We omit these conditions since the random measures we will consider in this thesis automatically satisfy them. See Theorem 1.8 p.66-67 in [Jacod and Shiryaev, 2003] for the details.

properties which we denote $\nu_S$ and we call the *compensator of the jump measure*. Then, given any function $H$ on $\Omega \times \mathbb{R}_+ \times E$ which is $\mathcal{P} \otimes \mathcal{B}(E)$-measurable and for which $|H| * \mu_S$ is locally integrable, we can define the *compensated integral* (or *stochastic integral w.r.t. $\mu_S - \nu_S$*) by

$$H * (\mu_S - \nu_S) = H * \mu_S - H * \nu_S \tag{1.7}$$

which is a purely discontinuous local martingale.[15] An equivalent (but more cumbersome) notation is the following

$$H * (\mu_S - \nu_S)_t = \int_{0+}^{t} \int_{\mathbb{R}} H(\omega; s, x) \left(\mu_S(\omega; ds, dx) - \nu_S(\omega; ds, dx)\right), \;\; t \geq 0.$$

### 1.2.4 Characteristics and Canonical Representations of Semimartingales

Semimartingales can be described very precisely using a triplet $(B, C, \nu)$ consisting of some finite variation process $B$, the quadratic variation of the continuous martingale part $C$ and the compensator of the jump measure $\nu$. In this section, we define this triplet formally and give some properties that are related to it.

Consider some semimartingale $S = (S_t)_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$. Let $h : \mathbb{R} \to \mathbb{R}$ be a so-called *truncation function*, i.e. a function which is bounded and such that $h(x) = x$ is some neighbourhood of 0.[16] Define

$$\check{S}(h)_t = \sum_{0 < s \leq t} \left(\Delta S_s - h(\Delta S_s)\right), \;\; t \geq 0$$

and $S(h) = S - \check{S}(h)$. Since $\Delta S_s - h(\Delta S_s) \neq 0$ only if $|\Delta S_s| > a$ for some $a > 0$, we see that $S(h)$ has bounded (by $a$) jumps and that it is a special semimartingale (see Lemma 2.24 p.44 in [Jacod and Shiryaev, 2003]). Thus, we can write its canonical decomposition

$$S(h)_t = S_0 + B(h)_t + M(h)_t, \;\; t \geq 0,$$

[15] Note that the standard definition is different and we used, in fact, a useful defining property. See Definition 1.27 p.72 in [Jacod and Shiryaev, 2003].

[16] In latter chapters, we will generally choose $h(x) = x\mathbf{1}_{\{|x| \leq 1\}}$.

where $B(h)$ is a predictable finite variation process with $B(h)_0 = 0$ and $M(h)$ is a local martingale with $M(h)_0 = 0$. Using this decomposition, we can define the *characteristics* (or *characteristic triplet*) of $S$ as the triplet $(B, C, \nu)$ where $B = B(h)$, $C = \langle S^c \rangle$ and $\nu = \nu_S$ is the compensator of the jump measure $\mu_S$. Note that we can choose a version of $\nu$ such that process $(|x|^2 \wedge 1) * \nu$ is locally integrable (see 2.13 p.77 in [Jacod and Shiryaev, 2003]).

Given a semimartingale $S = (S_t)_{t \geq 0}$ with characteristics $(B, C, \nu)$ and jump measure $\mu_S$, we thus have the following representation

$$S_t = S_0 + B_t + S_t^c + h * (\mu_S - \nu)_t + \left((x - h(x)) * \mu_S\right)_t, \quad t \geq 0, \tag{1.8}$$

which is called the *canonical representation* of $S$.[17] (It is automatic that the compensated integral and the integral w.r.t. the jump measure are well-defined for any truncation function $h$.)

It is possible to simplify this expression even further when $S$ is a special semimartingale, which happens if and only if $(|x|^2 \wedge |x|) * \nu$ is locally integrable (see Proposition 2.29 p.82 in [Jacod and Shiryaev, 2003]). Then, if $S_t = S_0 + A_t + M_t$, $t \geq 0$, is its canonical decomposition, we have that the compensated integral is well-defined for the function $H(\omega; t, x) = x$ and

$$S_t = S_0 + A_t + S_t^c + x * (\mu_S - \nu)_t, \quad t \geq 0, \tag{1.9}$$

(see Corollary 2.38 p.85 in [Jacod and Shiryaev, 2003]).

### 1.2.5 Processes with Independent Increments (PII) and Lévy Processes

One of the most important and tractable examples of semimartingales is given by a class of processes with independent increments and, their stationary counterparts, the Lévy processes. For these processes, the characteristic triplet is deterministic, has a simple form and completely characterizes the law of the process. In this section, we give the basic definitions and some important results related to these processes.

[17] Note that we will often commit a slight abuse of notation by writing $f(x) * \mu$ for $f * \mu$.

**Definition 1.2.4** (Processes with independent increments (PII) and Lévy processes)**.** *A process with independent increments $S = (S_t)_{t\geq 0}$ (also abbreviated as PII) on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ is a càdlàg stochastic process adapted to $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ such that $S_0 = 0$ and the increments $S_t - S_s$ are independent of $\mathcal{F}_s$, for each $s \leq t$. A Lévy process $L = (L_t)_{t\geq 0}$ is a PII for which the distribution of the increments $L_t - L_s$, depends only on the difference $t - s$, for each $s \leq t$.*[18]

Note that a PII is not necessarily a semimartingale (see Section II.5 starting p.114 in [Jacod and Shiryaev, 2003] for more information), but in this thesis we will be interested only in PII that are also semimartingales. The following fact, which is given in Theorem 4.15 p.106 in [Jacod and Shiryaev, 2003], will be useful in our context.

**Proposition 1.2.5.** *Let $S = (S_t)_{t\geq 0}$ be a semimartingale with $S_0 = 0$. Then, $S$ is a PII if and only if there exists a version $(B, C, \nu)$ of its characteristics that are deterministic.*

In fact, as mentioned above, it is possible to go further and to show that the characteristic function of a PII semimartingale is completely determined by its characteristics (see Theorem 4.15 p.106 and Theorem 4.25 p.110 in [Jacod and Shiryaev, 2003]) but we will not use these facts. However the similar statement for Lévy processes will be used a lot (see Corollary 4.19 p.107 in [Jacod and Shiryaev, 2003] for the proof).

**Proposition 1.2.6** (Characteristics of Lévy processes and Lévy-Khintchine formula)**.** *A stochastic process $L = (L_t)_{t\geq 0}$ is a Lévy process if and only if $L$ is a semimartingale admitting a version of its characteristics $(B, C, \nu)$ (for the truncation function $h$) that has the form*

$$B_t = at, \;\; C_t = \sigma^2 t, \;\; \nu(\omega; dt, dx) = dtK(dx),$$

*where $a \in \mathbb{R}$, $\sigma^2 \geq 0$ and $K$ is a positive measure on $\mathbb{R}$ such that $K(\{0\}) = 0$ and*

$$\int_{\mathbb{R}} (|x|^2 \wedge 1) K(dx) < \infty.$$

[18]Note that in older texts (e.g. in [Jacod and Shiryaev, 2003]) the term *process with stationary independent increments* or *PIIS* is used rather than the now more common term of *Lévy process*.

*Moreover, for all $t \geq 0$ and $u \in \mathbb{R}$,*

$$\mathbf{E}(e^{iuL_t}) = \exp t\left(iua - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iuh(x))K(dx)\right).$$

In general, the exponent

$$\Psi(u) = iua - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iuh(x))K(dx)$$

is called the *Lévy-Khintchine exponent* of $L$. We interchangeably call $(a, \sigma^2, K)$ and $(B, C, \nu)$ the characteristic triplet of $L$. From the above proposition, we also see that a Lévy process is always a semimartingale. Thus, we can write its canonical representations (1.8) and (1.9) which in this case are called *Lévy-Itô decomposition.*

**Proposition 1.2.7.** *(Lévy-Itô decomposition) Let $L = (L_t)_{t\geq 0}$ be a Lévy process with characteristic triplet $(a, \sigma^2, K)$ (for the truncation function $h$). We have*

$$L_t = at + \sigma W_t + h * (\mu_L - \nu)_t + ((x - h(x)) * \mu_L)_t, \ \ t \geq 0, \tag{1.10}$$

*where $\mu_L$ is the jump measure, $W = (W_t)_{t\geq 0}$ is a standard Brownian motion and $\nu(\omega; ds, dx) = dsK(dx)$. Moreover, when $\int_{\mathbb{R}}(|x|^2 \wedge |x|)K(dx) < \infty$, (i.e. when $L$ is a special semimartingale) we have*

$$L_t = ct + \sigma W_t + x * (\mu_L - \nu)_t, \ \ t \geq 0, \tag{1.11}$$

*where*

$$c = a + \int_{\mathbb{R}} (x - h(x))K(dx).$$

An important comment is that, in the Lévy case, the processes in the decomposition (1.10) are independent. This is not necessarily true for the more general semimartingale decomposition. An other important fact is that Lévy processes do not jump at fixed times, i.e. we have $\Delta L_t = 0$ ($\mathbf{P} - a.s.$), for all $t \geq 0$.

Lévy processes also posses a duality property which relates them to a "time-reversed" version of the process. The following proposition is stated in Lemma 3.1 in [Carmona et al., 2001] and is an adaptation of the general result given in e.g. Proposition 41.8 p.287 in [Sato, 1999].

**Proposition 1.2.8** (Time-reversal of Lévy processes)**.** *Let $L = (L_t)_{t\geq 0}$ be a Lévy process. Then, for each $t > 0$, the process $(\tilde{L}_s)_{0\leq s\leq t}$ given by*

$$\tilde{L}_s = L_t - L_{(t-s)-},$$

*for all $0 \leq s \leq t$, has the same law as $(L_s)_{0\leq s\leq t}$.*

The last fact about Lévy processes that we will use extensively is the following law of large numbers. This result is well known (see e.g. Section 36 starting p.245 in [Sato, 1999]) but its proof relies, in general, on the approximation of Lévy processes by random walks. We give here another method based on the study of the Lévy-Itô decomposition.[19]

**Proposition 1.2.9** (Law of Large Numbers for Lévy Processes)**.** *Let $L = (L_t)_{t\geq 0}$ be a Lévy process with characteristic triplet $(a, \sigma^2, K)$ for the truncation function $h(x) = x\mathbf{1}_{\{|x|\leq 1\}}$. Assume that $\int_{\mathbb{R}}(|x|^2 \wedge |x|)K(dx) < \infty$ (or equivalently that $\mathbf{E}(|L_1|) < \infty$). Then, as $t \to \infty$,*

$$\frac{L_t}{t} \overset{a.s.}{\to} \mathbf{E}(L_1) = a + \int_{\{|x|>1\}} xK(dx).$$

*Proof.* The equivalence between $\mathbf{E}(|L_1|) < \infty$ and the condition on the measure $K$ follows from Theorem 25.3 p.159 in [Sato, 1999]. Then, using the Lévy-Itô decomposition (1.11), we obtain

$$\frac{L_t}{t} = \mathbf{E}(L_1) + \sigma\frac{W_t}{t} + \frac{x * (\mu_L - \nu)_t}{t}, \;\; t \geq 0.$$

But, as well known, $\frac{W_t}{t} \overset{a.s.}{\to} 0$, as $t \to \infty$. Now let $M = x * (\mu_L - \nu)$. We will show that

$$\frac{M_t}{t} \overset{a.s.}{\to} 0. \tag{1.12}$$

Note that, for all $t \geq 0$, using (1.7), we obtain

$$\begin{aligned} M_t &= M_t^{(1)} + M_t^{(2)} + M_t^{(3)} \\ &= \int_0^t \int_{\{|x|\leq 1\}} x\,(\mu_L - \nu)\,(ds, dx) + \int_0^t \int_{\{|x|>1\}} x\mu^L(ds, dx) \\ &- \int_0^t \int_{\{|x|>1\}} xdsK(dx). \end{aligned}$$

[19] This proof also appears in [Spielmann, 2018].

By Theorem 9 p.142 in [Liptser and Shiryayev, 1989], to prove that

$$\frac{M_t^{(1)}}{t} \overset{a.s.}{\to} 0, \tag{1.13}$$

it is enough that $\tilde{D}_\infty < +\infty$ ($\mathbf{P} - a.s.$), where $\tilde{D}$ is the compensator of the process[20] $D = (D_t)_{t\geq 0}$ defined by

$$D_t = \sum_{0<s\leq t} \frac{(\Delta M_s^{(1)}/(1+s))^2}{1 + |\Delta M_s^{(1)}/(1+s)|}, \text{ for all } t \geq 0,$$

and where $\Delta M_s^{(1)}$ are the jumps of $M^{(1)}$. Since $\nu(\{s\}, dx) = \lambda(\{s\})K(dx) = 0$, because $\lambda$ is the Lebesgue measure, we obtain $\Delta M_s^{(1)} = \Delta L_s \mathbf{1}_{\{|\Delta L_s| \leq 1\}}$ by Theorem 1 p.176 in [Liptser and Shiryayev, 1989]. Thus, for all $t \geq 0$,

$$D_t = \sum_{0<s\leq t} \frac{(\Delta L_s)^2 \mathbf{1}_{\{|\Delta L_s|\leq 1\}}/(1+s)}{1+s+|\Delta L_s|\mathbf{1}_{\{|\Delta L_s|\leq 1\}}} = \int_0^t \int_{\mathbb{R}} \frac{x^2 \mathbf{1}_{\{|x|\leq 1\}}/(1+s)}{1+s+|x|\mathbf{1}_{\{|x|\leq 1\}}} \mu_L(ds, dx).$$

Therefore, $\tilde{D}$ satisfies

$$\begin{aligned}
\tilde{D}_t &= \int_0^t \int_{\mathbb{R}} \frac{|x|^2 \mathbf{1}_{\{|x|\leq 1\}}/(1+s)}{1+s+|x|\mathbf{1}_{\{|x|\leq 1\}}} ds K(dx) \\
&\leq \left( \int_0^t \frac{1}{(1+s)^2} ds \right) \left( \int_{\{|x|\leq 1\}} |x|^2 K(dx) \right) \\
&\leq \left( \int_0^\infty \frac{1}{(1+s)^2} ds \right) \left( \int_{\{|x|\leq 1\}} |x|^2 K(dx) \right) = \int_{\{|x|\leq 1\}} |x|^2 K(dx) < \infty,
\end{aligned}$$

for all $t \geq 0$, where the last integral is finite by definition of the measure $K$. So, $\tilde{D}_\infty < \infty$ ($\mathbf{P} - a.s.$), so (1.13) holds and if $K(\{|x| > 1\}) = 0$, (1.12) also holds.

Therefore, without loss of generality, we suppose that $K(\{|x| > 1\}) > 0$. Note that $\frac{M_t^{(3)}}{t} = -\int_{\{|x|>1\}} xK(dx)$, for all $t \geq 0$, so to complete the proof

[20] Up to now, we had only defined the compensator of a random measure. The general definition is as follows: the compensator $\tilde{A} = (\tilde{A})_{t\geq 0}$ of an increasing locally integrable process $A = (A_t)_{t\geq 0}$ is the unique increasing locally integrable predictable process such that $(A_t - \tilde{A}_t)_{t\geq 0}$ is a local martingale. See e.g. Theorem 3 p.33 in [Liptser and Shiryayev, 1989] for the proof of the existence of this object.

we need to show that

$$\frac{M_t^{(2)}}{t} \overset{a.s.}{\to} \int_{\{|x|\leq 1\}} xK(dx). \tag{1.14}$$

It is well known that the jump measure $\mu_L$ of a Lévy process is a Poisson random measure with intensity $\lambda \times K$, where $\lambda$ is the Lebesgue measure (see e.g. Proposition 3.7 p.79 in [Cont and Tankov, 2004]). Then, by Lemma 2.8 p.46-47 in [Kyprianou, 2014], $M^{(2)}$ is a compound Poisson process with rate $K(\{|x|>1\})$ and jump distribution $K(\{|x|>1\})^{-1}K(dx)|_{\{|x|>1\}}$ (where $K(dx)|_{\{|x|>1\}}$ represents the restriction of the measure $K$ to the set $\{|x|>1\}$). More precisely, we can write

$$M_t^{(2)} = \sum_{i=1}^{N_t} Y_i, \ \ t \geq 0,$$

where $N = (N_t)_{t\geq 0}$ is a Poisson process with rate $K(\{|x|>1\})$ and $(Y_i)_{i\in\mathbb{N}^*}$ is a sequence of i.i.d. random variables, which is independent from $N$ and with distribution $K(\{|x|>1\})^{-1}K(dx)|_{\{|x|>1\}}$. Conditioning on $N_t$, using the strong law of large numbers and noting that $N_t \overset{a.s.}{\to} +\infty$, we obtain

$$\frac{M_t^{(2)}}{N_t} = \frac{1}{N_t}\sum_{i=1}^{N_t} Y_i \overset{a.s.}{\to} \mathbf{E}(Y_1) = K(|x|>1)^{-1}\int_{\{|x|>1\}} xK(dx).$$

Finally, using the fact that $\frac{N_t}{t} \overset{a.s.}{\to} K(\{|x|>1\})$, we obtain

$$\frac{M_t^{(2)}}{t} = \frac{N_t}{t}\frac{M_t^{(2)}}{N_t} \overset{a.s.}{\to} \int_{\{|x|>1\}} xK(dx),$$

as $t \to \infty$, and so (1.14) holds, which implies (1.12). □

### 1.2.6 Doléans-Dade Exponential and Exponential Transform

As we will see, the main objects of this thesis, the generalized Ornstein-Uhlenbeck processes, can be seen as the solution of certain stochastic differential equations. Thus, we will need some stochastic analogue of the exponential function which we define in this section. The following specialization of Theorem 4.61 p.59 in [Jacod and Shiryaev, 2003] defines this object.

**Proposition 1.2.10.** *(Doléans-Dade Exponential) Let* $S = (S_t)_{t \geq 0}$ *be a semimartingale. The equation*

$$dY_t = Y_{t-} dS_t, \;\; t \geq 0,$$

$Y_0 = 1$ *admits a unique (up to indistinguishability) càdlàg adapted solution. This solution, which is denoted* $\mathcal{E}(S) = (\mathcal{E}(S)_t)_{t \geq 0}$ *and called the Doléans-Dade or stochastic exponential, is a semimartingale and is given explicitly by*

$$\mathcal{E}(S)_t = \exp\left(S_t - \frac{1}{2}\langle S^c \rangle_t\right) \prod_{0<s\leq t} (1 + \Delta S_s) e^{-\Delta S_s}, \;\; t \geq 0.$$

*Furthermore,*

1. *when* $S$ *is a local martingale, then* $\mathcal{E}(S)$ *is a local martingale and*
2. *we have* $\mathcal{E}(S) \neq 0$ *on the stochastic interval* $[\![0, \tau[\![$ *and* $\mathcal{E}(S) = 0$ *on* $[\![\tau, \infty[\![$*, where* $\tau = \inf\{t \geq 0 : \Delta S_t = -1\}$*, with* $\inf\{\emptyset\} = \infty$.

Actually, we have the following slightly more general result.

**Remark 1.2.11.** We can replace the initial condition $Y_0 = 1$ in the SDE of Proposition 1.2.10 by $Y_0 = Z$ where $Z$ is any $\mathcal{F}_0$-measurable random variable. Then, the claimed result still holds and the solution is given by

$$\mathcal{E}(S)_t = Z \exp\left(S_t - \frac{1}{2}\langle S^c \rangle_t\right) \prod_{0<s\leq t} (1 + \Delta S_s) e^{-\Delta S_s}, \;\; t \geq 0.$$

See Théorème 6.2 p.190 in [Jacod, 1979] for a proof.

We see that when $\Delta S_t > -1$, for all $t \geq 0$, we have also $\mathcal{E}(S)_t > 0$, for all $t \geq 0$. In that case, we can define its *exponential transform* as the process $\hat{S} = (\hat{S}_t)_{t \geq 0}$ given by $\hat{S}_t = \ln(\mathcal{E}(S)_t)$, i.e. the process such that $\exp \hat{S}_t = \mathcal{E}(S)_t$, for all $t \geq 0$. It is easy to see that $\hat{S}$ is again a semimartingale and that

$$\hat{S}_t = S_t - \frac{1}{2}\langle S^c \rangle_t + \sum_{0<s\leq t} (\ln(1 + \Delta S_s) - \Delta S_s),$$

because, rewriting Doléans-Dade's exponential, we have

$$\mathcal{E}(S)_t = \exp\left( S_t - \frac{1}{2}\langle S^c \rangle_t + \sum_{0<s\leq t} (\ln(1+\Delta S_s) - \Delta S_s) \right).$$

The jumps of $\hat{S}$ are given by

$$\Delta \hat{S}_t = \ln(1+\Delta S_t), \ \ t \geq 0.$$

Moreover, it is possible to check that when $L$ is a Lévy process, its exponential transform $\hat{L}$ is also a Lévy process (see Corollary 8.16 p.137 in [Jacod and Shiryaev, 2003]). In fact, it is possible to go a bit further and obtain the characteristics of the exponential transform from the initial process and vice-versa (see Theorem 8.10 p.136 in [Jacod and Shiryaev, 2003] for the proof of the following proposition).

**Proposition 1.2.12.** *(Characteristics of the Exponential Transform) Let $S = (S_t)_{t\geq 0}$ be a semimartingale with $\Delta S_t > -1$, for all $t \geq 0$. Let $\hat{S} = (\hat{S}_t)_{t\geq 0}$ be its exponential transform. Denote by $(B, C, \nu_S)$, respectively $(\hat{B}, \hat{C}, \nu_{\hat{S}})$, the characteristics of $S$ for the truncation function $h$, respectively of $\hat{S}$ for the truncation function $\hat{h}$. Then,*

$$\begin{cases} B = \hat{B} + \frac{\hat{C}}{2} + \Big( h(e^x - 1) - \hat{h}(x) \Big) * \nu_{\hat{S}} \\ C = \hat{C} \\ \mathbf{1}_{\{x\in G\}} * \nu_S = \mathbf{1}_{\{(e^x-1)\in G\}} * \nu_{\hat{S}} \end{cases}$$

*and*

$$\begin{cases} \hat{B} = B - \frac{C}{2} + \Big( \hat{h}(\ln(1+x)) - h(x) \Big) * \nu_S \\ \hat{C} = C \\ \mathbf{1}_{\{x\in G\}} * \nu_{\hat{S}} = \mathbf{1}_{\{\ln(1+x)\in G\}} * \nu_S, \end{cases}$$

*for any set $G \in \mathcal{B}(\mathbb{R})$.*

## 1.3 Generalized Ornstein-Uhlenbeck (GOU) processes

In this section, we define the Generalized Ornstein-Uhlenbeck or GOU processes which will be the main objects of this thesis. We start with a brief review of the classical Ornstein-Uhlenbeck processes and its generalizations.

The Ornstein-Uhlenbeck (OU) process was initially introduced as model for the motion of a particle in a fluid which is subjected a frictional force, see the classic paper [Uhlenbeck and Ornstein, 1930]. Rescaling the physical parameters, it can be defined by the following SDE:

$$dY_t = -\lambda Y_t dt + dB_t, \ \ t \geq 0, \tag{1.15}$$

where $Y_0 = y \in \mathbb{R}$ is the starting point of the process, $\lambda > 0$ and $B = (B_t)_{t\geq 0}$ is a standard Brownian motion. This SDE can easily be solved to obtain the explicit expression

$$Y_t = e^{-\lambda t}\left(y + \int_0^t e^{-\lambda s} dB_s\right) = ye^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} dB_s, \ \ t \geq 0. \tag{1.16}$$

Since it has been introduced, the OU process has been used to model numerous phenomena that exhibited the so-called *mean-reverting property*, which essentially means that the phenomenon has a tendency to revert to some mean state over time. For example, it has famously been suggested in [Vasicek, 1977] as a model for the evolution of interest rates.

From a mathematical point of view, a first generalization of the OU process appears in [Hadjiev, 1985]. In this first generalization, the Brownian motion $B$ is replaced with a more general Lévy process $L = (L_t)_{t\geq 0}$, so that the considered equation becomes

$$Y_t = -\lambda Y_t dt + dL_t, \ \ t \geq 0, \tag{1.17}$$

with a similar explicit form. Such processes are called *Lévy-driven* or *Lévy-type Ornstein-Uhlenbeck (LOU) processes*[21]. Again, these processes were used in mathematical finance as models for the interest rates, see [Patie, 2005].

We see, in this first generalization of the OU process, that the idea is to replace the Brownian motion $B$ appearing in (1.15) by a more general process. In a similar fashion, we could also try to replace $-\lambda dt$ by something more

[21]Note that some authors, see e.g. [Hadjiev, 1985], used the term *generalized Ornstein-Uhlenbeck processes* for these processes. However, since they are not the most general possible, this terminology is less natural.

general. In fact, this line of inquiry leads naturally to the study of the following linear stochastic differential equation

$$dY_t = dX_t + Y_{t-}dR_t, \;\; t \geq 0, \tag{1.18}$$

where $Y_0$ is some $\mathcal{F}_0$-measurable random variable, and $X = (X_t)_{t\geq 0}$ and $R = (R_t)_{t\geq 0}$ are two semimartingales. We see that, if $R_t = -\lambda t$ and $X_t = B_t$, we obtain the OU process and, if $R$ is as before and $X_t = L_t$, we obtain the LOU process. It is in that sense that (1.18) is a generalized version of (1.15) and (1.17). Note also that when $X_t = 0$ and $Y_0 = 1$, we obtain the Doléans-Dade exponential.

### 1.3.1 Existence and Uniqueness of Solutions for Linear SDEs

The question of the existence and uniqueness of the solutions of (1.18) is settled in the following proposition.

**Proposition 1.3.1** (Théorème 6.8 p.194 in [Jacod, 1979]). *Define the following sequence of stopping times $\tau_0 = 0$ and $\tau_{n+1} = \inf\{t > \tau_n | \Delta R_t = -1\}$, for all $n \in \mathbb{N}$, with $\inf\{\emptyset\} = \infty$. Then, the SDE (1.18) with the initial condition $Y_0 = y \in \mathbb{R}$ admits an unique (up to indistinguishability) solution given by $Y_t = \sum_{n\in\mathbb{N}} Y_t^{(n)} \mathbf{1}_{\{(t,.)\in[\![\tau_n,\tau_{n+1}[\![\}}$, with*

$$\begin{aligned} Y_t^{(n)} = U_t^{(n)} \Bigg( y + \Delta X_{\tau_n} &+ \int_{0+}^t (U_{s-}^{(n)})^{-1} d(X_s^{\tau_{n+1}} - X_s^{\tau_n}) \\ &- \int_{0+}^t (U_s^{(n)})^{-1} \mathbf{1}_{\{(s,.)\in[\![0,\tau_{n+1}[\![\}} d[X, R^{\tau_{n+1}} - R^{\tau_n}]_s \Bigg) \end{aligned} \tag{1.19}$$

*where $U_t^{(n)} = \mathcal{E}(R^{\tau_{n+1}} - R^{\tau_n})_t$, $t \geq 0$.*

*Proof.* To prove the uniqueness, let $Y = (Y_t)_{t\geq 0}$ and $\tilde{Y} = (\tilde{Y}_t)_{t\geq 0}$ be two solutions of (1.18). Then, $Z_t = Y_t - \tilde{Y}_t$ solves

$$dZ_t = Z_{t-}dR_t, \;\; t \geq 0,$$

with $Z_0 = 0$, and thus, by Remark 1.2.11, $Y$ is indistinguishable from $\tilde{Y}$.

We now show that the semimartingale defined in Equation (1.19) solves (1.18). For simplicity, write $X^{(n)} = X^{\tau_{n+1}} - X^{\tau_n}$ and $R^{(n)} = R^{\tau_{n+1}} - R^{\tau_n}$. Define

$$H_t^{(n)} = (y + \Delta X_{\tau_n})\mathbf{1}_{\{(t,.)\in[\![\tau_n,\infty[\![\}} + \int_{0+}^{t} (U_{s-}^{(n)})^{-1} dX_s^{(n)} \\ - \int_{0+}^{t} (U_s^{(n)})^{-1}\mathbf{1}_{\{(s,.)\in[\![0,\tau_{n+1}]\!]\}} d[X, R^{(n)}]_s, \;\; t \geq 0.$$

Letting $\tilde{Y}^{(n)} = U^{(n)}H^{(n)}$, we see that $\tilde{Y}^{(n)} = Y^{(n)} = Y$ on the set $[\![\tau_n, \tau_{n+1}[\![$ and $\tilde{Y}_-^{(n)} = Y_-^{(n)} = Y_-$ on the set $]\!]\tau_n, \tau_{n+1}]\!]$.

Taking the difference between $U_t^{(n)}H_t^{(n)}$ and $U_{\tau_n}^{(n)}H_{\tau_n}^{(n)}$, and using the definitions of the square brackets (1.3), we obtain, for all $t \geq 0$,

$$\tilde{Y}_t^{(n)} = (y + \Delta X_{\tau_n})\mathbf{1}_{\{(t,.)\in[\![\tau_n,\infty[\![\}} + \int_{\tau_n+}^{t} H_{s-}^{(n)} dU_s^{(n)} + \int_{\tau_n+}^{t} U_{s-}^{(n)} dH_s^{(n)} \\ + [H^{(n)}, U^{(n)}]_t - [H^{(n)}, U^{(n)}]_{\tau_n}.$$

But, $[H^{(n)}, U^{(n)}]_{\tau_n} = 0$. Moreover, $U^{(n)}$ and $H^{(n)}$ are constant on $[\![0, \tau_n]\!]$, so

$$\int_{\tau_n+}^{t} H_{s-}^{(n)} dU_s^{(n)} = \int_{0+}^{t} H_{s-}^{(n)} dU_s^{(n)}$$

and

$$\int_{\tau_n+}^{t} U_{s-}^{(n)} dH_s^{(n)} = \int_{0+}^{t} U_{s-}^{(n)} dH_s^{(n)}.$$

This yields, for all $t \geq 0$,

$$\tilde{Y}_t^{(n)} = (y + \Delta X_{\tau_n})\mathbf{1}_{\{(t,.)\in[\![\tau_n,\infty[\![\}} + \int_{0+}^{t} H_{s-}^{(n)} dU_s^{(n)} + \int_{0+}^{t} U_{s-}^{(n)} dH_s^{(n)} \\ + [H^{(n)}, U^{(n)}]_t.$$

Now we simplify the different terms appearing on the r.h.s. of the equation above. First, we have $dU_t^{(n)} = U_{t-}^{(n)} dR_t^{(n)}$, $t \geq 0$, and so

$$\int_{0+}^{t} H_{s-}^{(n)} dU_s^{(n)} = \int_{0+}^{t} H_{s-}^{(n)} U_{s-}^{(n)} dR_s^{(n)} = \int_{0+}^{t} \tilde{Y}_{s-}^{(n)} dR_s^{(n)}. \qquad (1.20)$$

Then, we have

$$\int_{0+}^{t} U_{s-}^{(n)} dH_s^{(n)} = X_t^{(n)} - \int_{0+}^{t} \frac{U_{s-}^{(n)}}{U_s^{(n)}} \mathbf{1}_{\{(s,.)\in [\![0,\tau_{n+1}[\![\}} d[X, R^{(n)}]_s. \tag{1.21}$$

Next, we have

$$\begin{aligned}
[H^{(n)}, U^{(n)}]_t &= \int_{0+}^{t} U_{s-}^{(n)} d[H^{(n)}, R^{(n)}]_s \\
&= [X^{(n)}, R^{(n)}]_t - \int_{0+}^{t} \frac{U_{s-}^{(n)}}{U_s^{(n)}} \mathbf{1}_{\{(s,.)\in [\![0,\tau_{n+1}[\![\}} d[[X, R^{(n)}], R^{(n)}]_s \\
&= [X^{(n)}, R^{(n)}]_t - \int_{0+}^{t} \frac{U_{s-}^{(n)} \Delta R_s^{(n)}}{U_s^{(n)}} \mathbf{1}_{\{(s,.)\in [\![0,\tau_{n+1}[\![\}} d[X, R^{(n)}]_s
\end{aligned} \tag{1.22}$$

since

$$[[X, R^{(n)}], R^{(n)}]_t = \int_{0+}^{t} \Delta R_{s-}^{(n)} d[X, R^{(n)}]_s,$$

by Proposition 4.49 p.52 in [Jacod and Shiryaev, 2003].

Now, since $\Delta U_t^{(n)} = U_{t-}^{(n)} \Delta R_t^{(n)}$, for all $t \geq 0$, we obtain

$$\begin{aligned}
&\int_{0+}^{t} \frac{U_{s-}^{(n)}}{U_s^{(n)}} \mathbf{1}_{\{(s,.)\in [\![0,\tau_{n+1}[\![\}} d[X, R^{(n)}]_s + \int_{0+}^{t} \frac{U_{s-}^{(n)} \Delta R_s^{(n)}}{U_s^{(n)}} \mathbf{1}_{\{(s,.)\in [\![0,\tau_{n+1}[\![\}} d[X, R^{(n)}]_s \\
&\quad = \int_{0+}^{t} \left( \frac{U_{s-}^{(n)}}{U_s^{(n)}} + \frac{U_{s-}^{(n)} \Delta R_s^{(n)}}{U_s^{(n)}} \right) \mathbf{1}_{\{(s,.)\in [\![0,\tau_{n+1}[\![\}} d[X, R^{(n)}]_s \\
&\quad = \int_{0+}^{t} \left( \frac{U_{s-}^{(n)}}{U_s^{(n)}} + \frac{\Delta U_s^{(n)}}{U_s^{(n)}} \right) \mathbf{1}_{\{(s,.)\in [\![0,\tau_{n+1}[\![\}} d[X, R^{(n)}]_s \\
&\quad = [X, R^{(n)}]_{t\wedge(\tau_{n+1})-}.
\end{aligned}$$

Thus, putting the above computation and (1.20), (1.21) and (1.22) together, we obtain

$$\begin{aligned}
\tilde{Y}_t^{(n)} = (y + \Delta X_{\tau_n}) \mathbf{1}_{\{(t,.)\in [\![\tau_n,\infty[\![\}} + X_t^{(n)} + \int_{0+}^{t} \tilde{Y}_{s-}^{(n)} dR_s^{(n)} \\
+ [X^{(n)}, R^{(n)}]_t - [X, R^{(n)}]_{t\wedge(\tau_{n+1})-}.
\end{aligned}$$

Now, using Theorem 23 p.68 part (iii) in [Protter, 2005], we obtain

$$[X^{(n)}, R^{(n)}]_t = [X^{\tau_{n+1}}, R^{(n)}]_t - [X^{\tau_n}, R^{(n)}]_t = [X, R^{(n)}]_{t\wedge\tau_{n+1}}.$$

And so, using Theorem 23 p.68 part (i) in [Protter, 2005] and the fact that $\Delta R^{(n)}_{\tau_{n+1}} = -1$, we obtain on the set $\{\tau_{n+1} < \infty\}$,

$$\begin{aligned}
[X^{(n)}, R^{(n)}]_t - [X, R^{(n)}]_{t\wedge(\tau_{n+1})-} &= \Delta[X, R^{(n)}]_{\tau_{n+1}} \mathbf{1}_{\{(t,.)\in[\![\tau_{n+1},\infty[\![\}} \\
&= \Delta X_{\tau_{n+1}} \Delta R^{(n)}_{\tau_{n+1}} \mathbf{1}_{\{(t,.)\in[\![\tau_{n+1},\infty[\![\}} \\
&= -\Delta X_{\tau_{n+1}} \mathbf{1}_{\{(t,.)\in[\![\tau_{n+1},\infty.[\![\}}
\end{aligned}$$

On the event $]\!]\tau_n, \tau_{n+1}]\!]$, we also have $\int_{0+}^t \tilde{Y}^{(n)}_{s-} dR^{(n)}_s = \int_{0+}^t Y^{(n)}_{s-} dR^{(n)}_s$, and thus we have shown that

$$\begin{aligned}
\tilde{Y}^{(n)}_t = y + X^{(n)}_t + \int_{0+}^t Y^{(n)}_{s-} dR^{(n)}_s + \Delta X_{\tau_n} \mathbf{1}_{\{(t,.)\in[\![\tau_n,\infty[\![\}} \\
- \Delta X_{\tau_{n+1}} \mathbf{1}_{\{(t,.)\in[\![\tau_{n+1},\infty[\![\}}.
\end{aligned} \tag{1.23}$$

Next, note that

$$Y^{\tau_{n+1}}_t - Y^{\tau_n}_t = \begin{cases} 0 & \text{on } [\![0, \tau_n[\![ \\ Y_t - Y_{\tau_n} & \text{on } [\![\tau_n, \tau_{n+1}[\![ \\ Y_{\tau_{n+1}} - Y_{\tau_n} & \text{on } [\![\tau_{n+1}, \infty[\![. \end{cases}$$

Since $Y^{(n)} = 0$ on $[\![0, \tau_n[\![$ and $Y = \tilde{Y}^{(n)} = (\tilde{Y}^{(n)})^{\tau_{n+1}}$ on $[\![\tau_n, \tau_{n+1}[\![$ with $\tilde{Y}^{(n)}_{\tau_{n+1}} = 0$, this is equivalent to

$$Y^{\tau_{n+1}}_t - Y^{\tau_n}_t = \tilde{Y}^{(n)}_t - Y_{\tau_n} \mathbf{1}_{\{(t,.)\in[\![\tau_n,\infty[\![\}} + Y_{\tau_{n+1}} \mathbf{1}_{\{(t,.)\in[\![\tau_{n+1},\infty[\![\}},$$

for al $t \geq 0$. Since $Y_{\tau_n} = \Delta X_{\tau_n}$ on $\{\tau_n < \infty\}$ and $Y_{\tau_{n+1}} = \Delta X_{\tau_{n+1}}$ on $\{\tau_{n+1} < \infty\}$, we obtain using (1.23), that

$$Y^{\tau_{n+1}}_t - Y^{\tau_n}_t = y + X^{(n)}_t + \int_{0+}^t Y_{s-} dR^{(n)}_s, \;\; t \geq 0.$$

Note that $\tau_n \to \infty$, as $n \to \infty$, and so we can sum the preceding expression over $n \in \mathbb{N}$ and we obtain a finite expression for $Y_t$ at each time $t \geq 0$. Moreover, $Y$ solves (1.18) by construction. $\square$

We see that a sequence of stopping times appears to avoid that the Doléans-Dade exponential of $R$ becomes 0. When the jumps of $R$ avoid $-1$, this problem doesn't occur and we have $\tau_0 = 0$ and $\tau_1 = \infty$ ($\mathbf{P} - a.s.$), which yields the following useful corollary.

**Corollary 1.3.2.** *Assume that $\tau_1 = \infty$ ($\mathbf{P}-a.s.$). Then, the SDE (1.18) with the initial condition $Y_0 = y \in \mathbb{R}$ admits an unique (up to indistinguishability) solution given by*

$$Y_t = \mathcal{E}(R)_t \left( y + \int_{0+}^{t} \mathcal{E}(R)_{s-}^{-1} dX_s - \int_{0+}^{t} \mathcal{E}(R)_{s}^{-1} d[X,R]_s \right), \quad t \geq 0.$$

In fact, in the context of models for the surplus of an insurance company, $\mathcal{E}(R)$ represent the price of some asset and thus we will require it to be strictly positive, which is equivalent to the condition $\Delta R_t > -1$, for all $t \geq 0$ ($\mathbf{P} - a.s.$).

## 1.3.2 Definition and Relation with Linear SDEs

We see, from Corollary 1.3.2, that the solution of (1.18) starts to look like the explicit form of the classical OU process (1.16). In fact when, in addition to $\tau_1 = \infty$ ($\mathbf{P} - a.s.$), we have $[X,R]_t = 0$ ($\mathbf{P} - a.s.$), for all $t \geq 0$, the solution of (1.18) is given by

$$Y_t = \mathcal{E}(R)_t \left( y + \int_{0+}^{t} \mathcal{E}(R)_{s-}^{-1} dX_s \right) = e^{\hat{R}_t} \left( y + \int_{0+}^{t} e^{-\hat{R}_{s-}} dX_s \right), \tag{1.24}$$

for all $t \geq 0$, where $\hat{R}$ is the exponential transform of $R$. Due to its similarity with the classical OU process, we call this process, the *generalized Ornstein-Uhlenbeck (GOU) process* associated with $X$ and $R$. The following proposition shows that in many cases of interest the assumption that $[X,R]_t = 0$ ($\mathbf{P} - a.s.$), for all $t \geq 0$, is automatically satisfied.

**Proposition 1.3.3.** *Assume that $X$ and $R$ are two independent semimartingales and that at least one of the processes is a Lévy process. Then, $[X,R]_t = 0$ ($\mathbf{P} - a.s.$), for all $t \geq 0$.*

*Proof.* Without loss of generality, we assume that $X$ is a Lévy process. We show first that

$$\sum_{0<s\leq t} \Delta X_s \Delta R_s = 0 \;\; (\mathbf{P} - a.s.), \;\; t \geq 0.$$

Fix $t \geq 0$, since $X$ and $R$ are càdlàg adapted processes there exists two countable sequences $(\chi_i)_{i\in\mathbb{N}^*}$ and $(\eta_j)_{j\in\mathbb{N}^*}$ which exhaust the jumps on $(0,t]$ of $X$ and $R$ respectively, see Proposition I.1.32 p.8 in [Jacod and Shiryaev, 2003]. Since, the processes are independent these sequences are also independent. Then, note that

$$\begin{aligned}
\mathbf{P}\left(\sum_{0<s\leq t} \Delta X_s \Delta R_s \neq 0\right) &= \mathbf{P}\left(\bigcup_{i=1}^{\infty}\bigcup_{j=1}^{\infty}\{\chi_i = \eta_j\}\right) \\
&\leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mathbf{P}(\chi_i = \eta_j).
\end{aligned}$$

Since $X$ is a Lévy process, we have $\mathbf{P}(\Delta X_s \neq 0) = 0$ and so $\mathbf{P}(\chi_i = s) = 0$, for all $i \in \mathbb{N}^*$ and $s \in (0,t]$. Thus, we obtain by independence $\mathbf{P}(\chi_i = \eta_j | \eta_j = s) = 0$ and

$$\mathbf{P}(\chi_i = \eta_j) = \mathbf{E}[\mathbf{P}(\chi_i = \eta_j | \eta_j)] = 0,$$

for all $i, j \in \mathbb{N}^*$. This yields the claimed fact.

Thus, $[X,R]_t = \langle X^c, R^c \rangle_t$ $(\mathbf{P} - a.s.)$, for all $t \geq 0$, by Equation (1.4). But, a product of independent continuous local martingales (w.r.t. some filtration) is also a local martingale (w.r.t. the same filtration) (see e.g. Theorem 2.4 in [Cherny, 2006]) and thus the result follows from Corollary I.4.55 p.55 in [Jacod and Shiryaev, 2003]. □

# Chapter 2

# GOU Processes as Weak Limits of Discrete-time Processes

In this chapter[1], we prove that the GOU process can be seen as the weak limit when the time steps goes to 0 of a large class of discrete-time processes, which are classically used to model the surplus of insurance companies facing both insurance and market risks. In our opinion, this chapter is the most important part of the thesis since it gives a theoretical argument for the importance of the GOU process in ruin theory and, more generally, in applied probability.

The chapter is structured as follows: after introducing some notations, we point to the related results in Section 2.1 and we prove the main weak convergence theorem in Section 2.2. From this result, we also deduce the convergence in distribution of the ruin times in the same section. Then, we give sufficient conditions for the convergence of the ultimate ruin probability in Section 2.3 and of the moments in Section 2.4, under general conditions. We illustrate these results using examples from actuarial theory and mathematical finance.

Let $(\xi_k)_{k\in\mathbb{N}^*}$ and $(\rho_k)_{k\in\mathbb{N}^*}$ be two independent sequences of i.i.d. random variables, with $\rho_k > 0$ ($\mathbf{P}-a.s.$) for all $k \in \mathbb{N}^*$. The *autoregressive process of order 1 with random coefficients*, abbreviated RCA(1) or RCAR(1), see e.g.

[1] This chapter is based on joint work with Yuchao Dong in [Dong and Spielmann, 2019] which is accepted for publication in *Insurance: Mathematics and Economics*.

[Nicholls and Quinn, 1982], is given by

$$\theta_k = \xi_k + \theta_{k-1}\rho_k, k \in \mathbb{N}^*. \tag{2.1}$$

and $\theta_0 = y \in \mathbb{R}$. Such processes, which are also called *stochastic recurrence or difference equations*, appear frequently in applied probability. For example, it is suggested in [Anděl, 1976] that RCA(1) processes could be useful in problems related to hydrology, meteorology and biology. We also refer to [Vervaat, 1979] for a more exhaustive list of examples. In ruin theory, the RCA(1) process is a classic model for the surplus capital of an insurance company where $(\xi_k)_{k\in\mathbb{N}^*}$ represents a stream of random payments or income and $(\rho_k)_{k\in\mathbb{N}^*}$ represents the random rates of return from one period to the next, see for example [Nyrhinen, 1999], [Nyrhinen, 2001], [Nyrhinen, 2012] and [Tang and Tsitsiashvili, 2003].

In this chapter, we will prove the convergence of the process (2.1) when the timestep goes to 0 and under a suitable re-normalization to the generalized Ornstein-Uhlenbeck (GOU) process given by

$$Y_t = e^{R_t}\left(y + \int_{0+}^t e^{-R_{s-}}dX_s\right), t \geq 0, \tag{2.2}$$

where $R = (R_t)_{t\geq 0}$ and $X = (X_t)_{t\geq 0}$ are independent stable Lévy processes with drift.[2]

One of the main uses of weak convergence is to prove the convergence of certain functionals of the path of the processes to the functional of the limiting process and to use the value of the latter as an approximation for the former, when the steps between two payments and their absolute values are small. Motivated by ruin theory, we will use the weak convergence result to prove the convergence of the ultimate ruin probability and the moments.

In general, the solution of the linear equation (1.18) or its explicit form (1.24) are chosen as a model for insurance surplus processes with investment risk on an a priori basis. The ruin problem is then studied under the heading

[2]In our notation of Section 1.2.6, we denoted the basic process $R$ and its exponential transform by $\hat{R}$. Here, since the important process is the exponential transform, we denote it for simplicity by $R$ and we will use $\tilde{R}$ to denote the *inverse exponential transform* $\tilde{R}_t = \mathfrak{L}(e^{R_t})$, for $t \geq 0$, where $\mathfrak{L}$ is the *stochastic logarithm.* (See Section 8 starting p.134 in [Jacod and Shiryaev, 2003], for the definition and the properties of the stochastic logarithm.)

"ruin problem with investment" for different choices of $R$ and $X$. We refer to Section 3.1 for an overview of the relevant literature. The main convergence results of this chapter could thus also be seen as a theoretical justification for the continuous-time model (2.2) in the context of models for insurance surplus processes with both insurance and market risks in the spirit of [Duffie and Protter, 1992].

Before pointing to the related literature, we now state the Burkholder-Davis-Gundy (BDG) and Doob inequalities which we will be used later in this chapter for convenience.

**Proposition 2.0.1** (Burkholder-Davis-Gundy (BDG) inequality, see Theorem 6 p.70 and Theorem 7 p.75 in [Liptser and Shiryayev, 1989])**.** *Let $\tau$ be a stopping time (which can be $\infty$), $M = (M_t)_{t\geq 0}$ be a local martingale, with $M_0 = 0$ and $q \geq 1$. Then, there exists constants $c_q$ and $C_q$ (independent of $\tau$ and $M$) such that*

$$c_q \mathbf{E}\left([M]_\tau^{q/2}\right) \leq \mathbf{E}\left(\sup_{0\leq t\leq \tau} (M_t)^q\right) \leq C_q \mathbf{E}\left([M]_\tau^{q/2}\right).$$

The Doob inequality is usually stated for uniformly integrable martingales, but it can be adapted to non-negative submartingales.

**Proposition 2.0.2** (Doob's inequality, Theorem (2-2) p.44 in [Mémin, 1978])**.** *Let $T \in [0, \infty]$, $q > 1$ and $M = (M_t)_{t\geq 0}$ be a non-negative submartingale satisfying $\mathbf{E}(M_T^q) < \infty$. Then,*

$$\mathbf{E}\left(\sup_{0\leq t\leq T} (M_t)^q\right) \leq \left(\frac{q}{q-1}\right)^q \mathbf{E}(M_T^q).$$

## 2.1 Related Results

In actuarial mathematics, similar weak convergence results and approximations of functionals to the ones studied in this chapter are a well developed line of research. In [Iglehart, 1969] it is shown that the compound Poisson process with drift converges weakly to a Brownian motion with drift and it is shown that the finite-time and ultimate ruin probability converge to those of the limiting model. These results are extended to processes with more general jump times in [Grandell, 1977] and with more general jump sizes in [Burnecki, 2000] and [Furrer et al., 1997]. Similar convergence results are proven for the integral of a deterministc function w.r.t. a compound Poisson process in [Harrison, 1977], this corresponds to the assumption that the insurance company can invest at a deterministic interest rate. Some of the previous results are generalized in [Paulsen and Gjessing, 1997], where it is shown that a general model with a jump-diffusion surplus process and stochastic jump-diffusion investment converges to a particular diffusion process.

More closely related to our results are the papers [Cumberland and Sykes, 1982] and [Dufresne, 1989]. In [Cumberland and Sykes, 1982], it is shown that the AR(1) process (i.e. when the coefficients $\rho_k$ are deterministic and constant) converges weakly to a standard OU process. In [Dufresne, 1989], it is shown that when the variables $\xi_k$ are deterministic and satisfy some regularity conditions, we have a similar weak convergence result where the process $X$ in (2.2) is replaced by a deterministic function.

The results in [Duffie and Protter, 1992] are also closely related. In that paper, the authors study the weak convergence of certain discrete-time models to continuous-time models appearing in mathematical finance and prove the convergence of the values of certain functionals such as the call option price. In particular, for the case $\xi_k = 0$, for all $k \in \mathbb{N}^*$, they show, using the same re-normalization as we do (see below at the beginning of Section 2.2), that the discrete-time process (2.1) converges to the Doléans-Dade exponential of a Brownian motion with drift. This generalizes the famous paper [Cox et al., 1979] where it is shown that the exponential of a simple random walk correctly re-normalized converges to the Black-Scholes model.

Finally, the relationship between the discrete-time process (2.1) and (2.2) was also studied in [de Haan and Karandikar, 1989], where it is shown that GOU processes are continuous-time analogues of RCA(1) processes in some sense. More precisely, they show that any continuous-time process $S = (S_t)_{t\geq 0}$ for which the sequence $(S_{nh})_{n\in\mathbb{N}^*}$ of the process sampled at rate $h > 0$ satisfies an equation of the form (2.1), for all $h > 0$, with some additional conditions, is a GOU process of the form (2.2), where $X$ and $R$ are general Lévy processes. Our main result is coherent with this analogy but does not seem to be otherwise related.

We thus contribute to these results by treating the case where $(\xi_k)_{k\in\mathbb{N}^*}$ and

$(\rho_k)_{k\in\mathbb{N}^*}$ are sequences of non-degenerate random variables and by giving, in the square-integrable case, approximations of the ultimate ruin probability and the moments for the RCA(1) process.

## 2.2 Weak Convergence of the Processes and of the Ruin Times

In this section, we show that the discrete-time process correctly re-normalized converges weakly to the GOU process and prove the convergence in distribution of the ruin times. We introduce the following set of assumptions.

**Assumption ($\mathbf{H}^\alpha$).** We say that a random variable $Z$ satisfies ($\mathbf{H}^\alpha$) if its distribution function $F_Z$ satisfies

$$F_Z(-x) \sim k_1^Z x^{-\alpha} \text{ and } 1 - F_Z(x) \sim k_2^Z x^{-\alpha},$$

as $x \to \infty$, for some $1 < \alpha < 2$ and some constants $k_1^Z, k_2^Z$ such that $k_1^Z + k_2^Z > 0$. Note that this assumption implies that $\mathbf{E}(|Z|) < \infty$.

**Assumption ($\mathbf{H^2}$).** We say that a random variable $Z$ satisfies ($\mathbf{H^2}$) if $Z$ is square-integrable with $\mathrm{Var}(Z) > 0$, where $\mathrm{Var}(Z)$ is the variance of $Z$.

We now introduce some notations and recall some classical facts about weak convergence on metric spaces, stable random variables and Lévy processes.

Recall that the space $\mathbb{D}$ of càdlàg functions $\mathbb{R}_+ \to \mathbb{R}$ can be equipped with the Skorokhod metric which makes it a complete and separable metric space, see e.g. Section VI.1, p.324 in [Jacod and Shiryaev, 2003]. Let $\mathcal{D}$ be the Borel sigma-field for this topology. Given a sequence of random elements $Z^{(n)}$ : $(\Omega^{(n)}, \mathcal{F}^{(n)}, \mathbf{P}^{(n)}) \mapsto (\mathbb{D}, \mathcal{D})$, with $n \geq 1$, we say that $(Z^{(n)})_{n\geq 1}$ converges weakly or in distribution to $Z : (\Omega, \mathcal{F}, \mathbf{P}) \mapsto (\mathbb{D}, \mathcal{D})$, if the laws of $Z^{(n)}$ converge weakly to the law of $Z$, when $n \to \infty$. We denote weak convergence by $Z^{(n)} \xrightarrow{d} Z$ and we use the same notation for the weak convergence of measures on $\mathbb{R}$.

Concerning stable random variables $Z$ of index $\alpha$, the most common way to define them is trough their characteristic functions:

$$\mathbf{E}(e^{iuZ}) = \exp[i\gamma u - c|u|^\alpha(1 - i\beta \mathrm{sign}(u) z(u, \alpha))],$$

where $\gamma \in \mathbb{R}$, $c > 0$, $\alpha \in (0, 2]$, $\beta \in [-1, 1]$ and

$$z(u, \alpha) = \begin{cases} \tan\left(\frac{\pi\alpha}{2}\right) & \text{if } \alpha \neq 1, \\ -\frac{2}{\pi} \ln |u| & \text{if } \alpha = 1. \end{cases}$$

Stable Lévy processes $(L_t)_{t \geq 0}$ are Lévy processes such that $L_t$ is equal in law to some stable random variable, for each $t \geq 0$, with fixed parameters $\beta \in [-1, 1]$ and $\gamma = 0$ (see e.g. Definition 2.4.7 p.93 in [Embrechts et al., 1997].)

Finally, note that if $(Z_k)_{k \in \mathbb{N}^*}$ is a sequence of i.i.d. random variables such that $Z_1$ satisfies either $(\mathbf{H}^\alpha)$ or $(\mathbf{H^2})$, then there exists a stable random variable $K_\alpha$ and some constant $c_\alpha > 0$ such that

$$\sum_{k=1}^{n} \frac{Z_k - \mu_Z}{c_\alpha n^{1/\alpha}} \xrightarrow{d} K_\alpha, \tag{2.3}$$

as $n \to \infty$ where $\mu_Z = \mathbf{E}(Z_1)$. In fact, when $Z_1$ satisfies $(\mathbf{H^2})$, $\alpha = 2$, $c_\alpha = 1$ and $K_\alpha$ is the standard normal distribution with variance $\mathrm{Var}(Z_1)$. (See e.g. Section 2.2 p.70-81 in [Embrechts et al., 1997] for these facts.)

The main assumption we will use and which combines the previous ones is the following.

**Assumption (H).** We assume that $(\xi_k)_{k \in \mathbb{N}^*}$ and $(\rho_k)_{k \in \mathbb{N}^*}$ are two independent sequences of i.i.d. random variables, with $\rho_k > 0$ $(\mathbf{P} - a.s.)$ for all $k \in \mathbb{N}^*$, and that $\xi_1$ (resp. $\ln(\rho_1)$) satisfies either $(\mathbf{H}^\alpha)$ or $(\mathbf{H^2})$ (resp. $(\mathbf{H}^\beta)$ or $(\mathbf{H^2})$.) We denote by $c_\alpha$ (resp. $c_\beta$) the constant and by $K_\alpha$ (resp. $K_\beta$) the limiting stable random variable appearing in (2.3). In addition, we denote by $(L_t^\alpha)_{t \geq 0}$ (resp. $(L_t^\beta)_{t \geq 0}$) the stable Lévy processes obtained by putting $L_1^\alpha \stackrel{d}{=} K_\alpha$ (resp. $L_1^\beta \stackrel{d}{=} K_\beta$).

We can now state the main results of this section. Fix $n \in \mathbb{N}^*$, we want to divide the time interval into $n$ subintervals of length $1/n$ and update the discrete-time process (2.1) at each time point of the subdivision. To formalize this, we define the following process

$$\theta^{(n)}\left(\frac{k}{n}\right) = \xi_k^{(n)} + \theta^{(n)}\left(\frac{k-1}{n}\right)\rho_k^{(n)}, k \in \mathbb{N}^*, \tag{2.4}$$

where $(\xi_k^{(n)})_{k\in\mathbb{N}^*}$ and $(\rho_k^{(n)})_{k\in\mathbb{N}^*}$ have to be defined from the initial sequences. Following an idea in [Dufresne, 1989], we let $\mu_\xi = \mathbf{E}(\xi_1)$ and $\mu_\rho = \mathbf{E}(\ln(\rho_1))$ and define

$$\xi_k^{(n)} = \frac{\mu_\xi}{n} + \frac{\xi_k - \mu_\xi}{c_\alpha n^{1/\alpha}}$$

and $\rho_k^{(n)} = \exp(\gamma_k^{(n)})$ where

$$\gamma_k^{(n)} = \frac{\mu_\rho}{n} + \frac{\ln(\rho_k) - \mu_\rho}{c_\beta n^{1/\beta}}.$$

These definitions ensure that

$$\mathbf{E}\left(\sum_{k=1}^{n} \xi_k^{(n)}\right) = \mu_\xi \quad \text{and} \quad \mathbf{E}\left(\sum_{k=1}^{n} \ln(\rho_k^{(n)})\right) = \mu_\rho.$$

Moreover, when $\xi_1$ and $\ln(\rho_1)$ both satisfy $(\mathbf{H^2})$, we choose $\alpha = \beta = 2$ and $c_\alpha = c_\beta = 1$, and then we have the following variance stabilizing property:

$$\mathrm{Var}\left(\sum_{k=1}^{n} \xi_k^{(n)}\right) = \mathrm{Var}(\xi_1) \quad \text{and} \quad \mathrm{Var}\left(\sum_{k=1}^{n} \ln(\rho_k^{(n)})\right) = \mathrm{Var}(\ln(\rho_1)).$$

Finally, we define the filtrations $\mathcal{F}_0^{(n)} = \{\emptyset, \Omega\}$, $\mathcal{F}_k^{(n)} = \sigma((\xi_i^{(n)}, \rho_i^{(n)}), i = 1, \ldots, k)$, $k \in \mathbb{N}^*$ and $\mathcal{F}_t^{(n)} = \mathcal{F}_{[nt]}^{(n)}$, for $t \geq 0$, where $[.]$ is the floor function and define $\theta^{(n)}$ as the (continuous-time) stochastic process given by

$$\theta_t^{(n)} = \theta^{(n)}\left(\frac{[nt]}{n}\right), t \geq 0.$$

**Theorem 2.2.1.** *Under* $(\mathbf{H})$*, we have* $\theta^{(n)} \xrightarrow{d} Y$*, as* $n \to \infty$*, where* $Y = (Y_t)_{t\geq 0}$ *is the GOU process* (2.2) *with* $X_t = \mu_\xi t + L_t^\alpha$ *and* $R_t = \mu_\rho t + L_t^\beta$*, for all* $t \geq 0$*. In addition,* $Y$ *satisfies the following stochastic differential equation :*

$$Y_t = y + X_t + \int_{0+}^{t} Y_{s-} d\tilde{R}_s, t \geq 0, \tag{2.5}$$

*where*

$$\tilde{R}_t = R_t + \frac{1}{2}\langle R^c \rangle_t + \sum_{0<s\leq t} \left(e^{\Delta R_s} - 1 - \Delta R_s\right), t \geq 0,$$

*and* $R^c$ *is the continuous martingale part of* $R$ *and* $\Delta R_t$ *is its jump at time* $t \geq 0$*.*

**Example 2.2.2** (Pareto losses and stable log-returns)**.** The assumption ($\mathbf{H}^\alpha$) is quite general and simple to check. To illustrate it we take the negative of a Pareto (type I) distribution with shape parameter $1 < \alpha < 2$ for the loss $\xi_1$, i.e. the random variable defined by its distribution function $F_\xi(x) = (-x)^{-\alpha}$, for $x \leq -1$. The condition on $\alpha$ ensures that $\xi_1$ has a finite first moment, but an infinite second moment. Moreover, $\xi_1$ then satisfies ($\mathbf{H}^\alpha$), with constants $k_1^\xi = 1$ and $k_2^\xi = 0$. We also have that $\mu_\xi = -\alpha/(\alpha - 1)$ and that

$$\sum_{k=1}^{n} \frac{\xi_k - \mu_\xi}{c_{\alpha,\xi} n^{1/\alpha}} \xrightarrow{d} -K_{\alpha,\xi},$$

as $n \to \infty$, with

$$c_{\alpha,\xi} = \frac{\pi}{2\Gamma(\alpha)\sin(\alpha\pi/2)},$$

where $\Gamma$ is the Gamma function and where $K_{\alpha,\xi}$ is a stable random variable of index $\alpha$, with $\gamma = 0$, $c = 1$ and $\beta = 1$ (see e.g. p.62 in [Uchaikin and Zolotarev, 1999]).

For the log-returns $\ln(\rho_1)$, we take a stable distribution with index $1 < \tilde{\alpha} < 2$, and parameters $\tilde{\gamma} = 0$, $\tilde{c} = 1$ and $\tilde{\beta} \in [-1, 1]$. Then, we have $\mu_\rho = 0$ and

$$\sum_{k=1}^{n} \frac{\ln(\rho_k) - \mu_\rho}{c_{\tilde{\alpha},\rho} n^{1/\tilde{\alpha}}} \stackrel{d}{=} K_{\tilde{\alpha},\rho},$$

for all $n \in \mathbb{N}^*$. Thus, Theorem 2.2.1 implies that $\theta^{(n)} \xrightarrow{d} Y$, as $n \to \infty$, where

$$Y_t = e^{R_t}\left(y + \int_{0+}^{t} e^{-R_{s-}} dX_s\right), t \geq 0,$$

with $X_t = \mu_\xi t + L_t^\alpha$ and $R_t = \mu_\rho t + L_t^{\tilde{\alpha}}$, where $L^\alpha$ and $L^{\tilde{\alpha}}$ are stable Lévy processes with $L_1^\alpha \stackrel{d}{=} -K_{\alpha,\xi}$ and $L_1^{\tilde{\alpha}} \stackrel{d}{=} K_{\tilde{\alpha},\rho}$.

As already mentioned, we will be interested in the application of Theorem 2.2.1 to ruin theory and we now state the main consequence for this line of study. Define the following stopping times, for $n \geq 1$,

$$\tau^n(y) = \inf\{t > 0 : \theta_t^{(n)} < 0\}$$

with the convention $\inf\{\emptyset\} = +\infty$, and also

$$\tau(y) = \inf\{t > 0 : Y_t < 0\}.$$

**Theorem 2.2.3.** *Assume that* (**H**) *holds. We have, for all* $T \geq 0$,

$$\lim_{n\to\infty} \mathbf{P}(\tau^n(y) \leq T) = \mathbf{P}(\tau(y) \leq T)$$

*and, equivalently,* $\tau^n(y) \xrightarrow{d} \tau(y)$, *as* $n \to \infty$.

Theorem 2.2.3 implies the convergence of $\mathbf{E}(f(\tau^n(y))$ to $\mathbf{E}(f(\tau(y))$, for any continuous and bounded function $f : \mathbb{R}_+ \to \mathbb{R}$. For example, we can obtain the following convergence result for a simple form of the discounted penalty function.

**Corollary 2.2.4.** *Assume that* (**H**) *holds. We have*

$$\lim_{n\to\infty} \mathbf{E}(e^{-\alpha\tau^n(y)}\mathbf{1}_{\{\tau^n(y)<+\infty\}}) = \mathbf{E}(e^{-\alpha\tau(y)}\mathbf{1}_{\{\tau(y)<+\infty\}}),$$

*for all* $\alpha > 0$.

When $\xi_1$ and $\ln(\rho_1)$ both satisfy ($\mathbf{H^2}$), the limiting stable random variable is, in fact, the standard normal random variable and the limiting process is defined by two independent Brownian motions with drift.

**Corollary 2.2.5** (Pure diffusion limit)**.** *Assume that* $\xi_1$ *and* $\ln(\rho_1)$ *both satisfy* ($\mathbf{H^2}$), *then* $\theta^{(n)} \xrightarrow{d} Y$, *as* $n \to \infty$, *for* $Y = (Y_t)_{t\geq 0}$ *defined by* (2.2) *with* $R_t = \mu_\rho t + \sigma_\rho W_t$ *and* $X_t = \mu_\xi t + \sigma_\xi \tilde{W}_t$, *for all* $t \geq 0$, *where* $(W_t)_{t\geq 0}$ *and* $(\tilde{W}_t)_{t\geq 0}$ *are two independent standard Brownian motions and* $\sigma_\xi^2 = \mathrm{Var}(\xi_1)$ *and* $\sigma_\rho^2 = \mathrm{Var}(\ln(\rho_1))$.

**Example 2.2.6** (Pareto losses and NIG log-returns)**.** To illustrate ($\mathbf{H^2}$) we take again the negative of a Pareto (type I) distribution for the loss $\xi_1$ but with shape parameter $\alpha \geq 2$, so that the distribution admits also a second moment. For the log-returns, $\ln(\rho_1)$ we take the normal inverse gaussian $\mathrm{NIG}(\alpha, \beta, \delta, \mu)$ with parameters $0 \leq |\beta| < \alpha$, $\delta > 0$ and $\mu \in \mathbb{R}$, i.e. the random variable defined by the following moment generating function

$$\mathbf{E}(e^{u\ln(\rho_1)}) = \exp\left(\mu u + \delta\left(\gamma - \sqrt{\alpha^2 - (\beta+u)^2}\right)\right),$$

where $\gamma = \sqrt{\alpha^2 - \beta^2}$, for all $u \in \mathbb{R}$.

Then, it is well known that

$$\mu_\xi = -\frac{\alpha}{\alpha - 1}, \quad \sigma_\xi^2 = \frac{\alpha}{(\alpha-1)^2(\alpha-2)}$$

and that

$$\mu_\rho = \mu + \frac{\beta\delta}{\gamma}, \quad \sigma_\rho^2 = \delta\frac{\alpha^2}{\gamma^3}.$$

Thus, in this case, Corollary 2.2.5 yields $\theta^{(n)} \xrightarrow{d} Y$, with

$$Y_t = e^{\mu_\rho t + \sigma_\rho W_t}\left(y + \int_{0+}^t e^{-\mu_\rho s - \sigma_\rho W_s} d(\mu_\xi s + \sigma_\xi \tilde{W}_s)\right), t \geq 0,$$

and where $(W_t)_{t\geq 0}$ and $(\tilde{W}_t)_{t\geq 0}$ are two independent standard Brownian motions.

We now turn to the proofs of the previous theorems. The main strategy is to rewrite the discrete-time process as a stochastic integral and to use the well-known weak convergence result for stochastic integrals based on the UT (uniform tightness) condition for semimartingales.

To rewrite the discrete-time process, note that, by induction, the explicit solution of (2.4), for all $n \in \mathbb{N}^*$ and $k \in \mathbb{N}^*$, is given by

$$\begin{aligned}\theta^{(n)}\left(\frac{k}{n}\right) &= y\prod_{i=1}^k \rho_i^{(n)} + \sum_{i=1}^k \xi_i^{(n)} \prod_{j=i+1}^k \rho_j^{(n)} \\ &= \prod_{i=1}^k \rho_i^{(n)}\left(y + \sum_{i=1}^k \xi_i^{(n)} \prod_{j=1}^i (\rho_j^{(n)})^{-1}\right),\end{aligned}$$

where, by convention, we set $\prod_{j=k+1}^k \rho_j^{(n)} = 1$, for all $n \in \mathbb{N}^*$. Thus,

$$\theta_t^{(n)} = \prod_{i=1}^{[nt]} \rho_i^{(n)}\left(y + \sum_{i=1}^{[nt]} \xi_i^{(n)} \prod_{j=1}^i (\rho_j^{(n)})^{-1}\right). \tag{2.6}$$

and setting $X_t^{(n)} = \sum_{i=1}^{[nt]} \xi_i^{(n)}$ and $R_t^{(n)} = \sum_{i=1}^{[nt]} \gamma_i^{(n)}$, we obtain

$$\theta_t^{(n)} = e^{R_t^{(n)}}\left(y + \int_{0+}^t e^{-R_{s-}^{(n)}} dX_s^{(n)}\right). \tag{2.7}$$

In fact, the above rewriting of the discrete-time process will prove very useful for most proofs in this chapter.

**Remark 2.2.7.** An other way to prove the weak convergence would be to remark that since $[X^{(n)}, R^{(n)}]_t = 0$, for all $n \in \mathbb{N}^*$, we find that $\theta^{(n)}$ satisfies the following stochastic differential equation :

$$\theta_t^{(n)} = y + X_t^{(n)} + \int_{0+}^t \theta_{s-}^{(n)} d\tilde{R}_s^{(n)},$$

where

$$\tilde{R}_t^{(n)} = R^{(n)} + \sum_{0<s\leq t} (e^{\Delta R_s^{(n)}} - 1 - \Delta R_s^{(n)}) = \sum_{i=1}^{[nt]} (e^{\gamma_i^{(n)}} - 1),$$

and to use the well-known stability results for solutions of stochastic differential equations. We refer to [Duffie and Protter, 1992] for an interesting application of this method for different models in mathematical finance. However, this way seems harder, in our case, since the process $(\tilde{R}_t^{(n)})_{t\geq 0}$ is less explicit than $(R_t^{(n)})_{t\geq 0}$.

We now recall the UT condition, the weak convergence result and give two lemmas to check the condition in our case.

**Definition 2.2.8.** *Consider a sequence of real-valued semimartingales $Z^{(n)}$ defined on $(\Omega^{(n)}, \mathcal{F}^{(n)}, (\mathcal{F}_t^{(n)})_{t\geq 0}, \mathbf{P}^{(n)})$, for each $n \in \mathbb{N}^*$. Denote by $\mathcal{H}^{(n)}$ the set given by*

$$\begin{aligned} \mathcal{H}^{(n)} = \{H^{(n)} | H_t^{(n)} = L^{n,0} + \sum_{i=1}^{p} L^{n,i} \mathbf{1}_{[t_i, t_{i+1})}(t), p \in \mathbb{N}, \\ 0 = t_0 < t_1 < \cdots < t_p = t, \\ L^{n,i} \text{ is } \mathcal{F}_{t_i}^{(n)} - \text{measurable with } |L^{n,i}| \leq 1\}. \end{aligned}$$

*The sequence $(Z^{(n)})_{n\in\mathbb{N}^*}$ is UT (also called P-UT in [Jacod and Shiryaev, 2003], for "uniformly tight" and "predictably uniformly tight") if for all $t > 0$, for all $\epsilon > 0$, there exists $M > 0$ such that,*

$$\sup_{H^{(n)} \in \mathcal{H}^{(n)}, n\in\mathbb{N}^*} \mathbf{P}^{(n)} \left( \left| \int_{0+}^t H_{s-}^{(n)} dZ_s^{(n)} \right| > M \right) < \epsilon.$$

For more information see Section VI.6 in [Jacod and Shiryaev, 2003]. One of the interesting consequences of the UT condition is given by the following proposition which is a particular case of Theorem 6.22 p.383 of [Jacod and Shiryaev, 2003].

**Proposition 2.2.9.** *Let* $(H^{(n)}, Z^{(n)})_{n\in\mathbb{N}^*}$ *be a sequence of real-valued semimartingales defined on* $(\Omega^{(n)}, \mathcal{F}^{(n)}, (\mathcal{F}_t^{(n)})_{t\geq 0}, \mathbf{P}^{(n)})$*. If* $(H^{(n)}, Z^{(n)}) \xrightarrow{d} (H, Z)$ *as* $n \to \infty$ *and the sequence* $(Z^{(n)})_{n\in\mathbb{N}^*}$ *is UT, then* $Z$ *is a semimartingale and when* $n \to \infty$*,*

$$\left(H^{(n)}, Z^{(n)}, \int_0^{\cdot} H_{s-}^{(n)} dZ_s^{(n)}\right) \xrightarrow{d} \left(H, Z, \int_0^{\cdot} H_{s-} dZ_s\right).$$

The following lemma is based on Remark 6.6 p.377 in [Jacod and Shiryaev, 2003].

**Lemma 2.2.10.** *Let* $(Z^{(n)})_{n\in\mathbb{N}^*}$ *be a sequence of real-valued semimartingales with locally bounded variation defined on* $(\Omega^{(n)}, \mathcal{F}^{(n)}, (\mathcal{F}_t^{(n)})_{t\geq 0}, \mathbf{P}^{(n)})$*. If for each* $t > 0$ *and each* $\epsilon > 0$*, there exists* $M > 0$ *such that*

$$\sup_{n\in\mathbb{N}^*} \mathbf{P}^{(n)} \left(V(Z^{(n)})_t > M\right) < \epsilon,$$

*where* $V(.)$ *denotes the total first order variation of a process, then* $(Z^{(n)})_{n\geq 1}$ *is UT.*

*Proof.* For each $n \in \mathbb{N}^*$, $H^{(n)} \in \mathcal{H}^{(n)}$ and $t > 0$, we find $p \in \mathbb{N}$ and $0 = t_0 < t_1 < \cdots < t_p = t$ such that

$$\begin{aligned}\left|\int_{0+}^{t} H_{s-}^{(n)} dZ_s^{(n)}\right| &\leq |L^{n,0}| + \sum_{i=1}^{p} |L^{n,i}||Z_{t_{i+1}} - Z_{t_i}| \leq 1 + \sum_{i=1}^{p} |Z_{t_{i+1}} - Z_{t_i}| \\ &\leq 1 + V(Z^{(n)})_t.\end{aligned}$$

Thus, the assumption implies the UT property. □

The following lemma is based on Remark 2-1 in [Mémin and Słomiński, 1991].

**Lemma 2.2.11.** *Let* $(Z^{(n)})_{n\in\mathbb{N}^*}$ *be a sequence of real-valued local martingales defined on* $(\Omega^{(n)}, \mathcal{F}^{(n)}, (\mathcal{F}_t^{(n)})_{t\geq 0}, \mathbf{P}^{(n)})$ *and* $Z$ *a real-valued semimartingale on* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$*. Denote by* $\nu^{(n)}$ *the compensator of the jump measure of* $Z^{(n)}$*. If* $Z^{(n)} \xrightarrow{d} Z$ *as* $n \to \infty$*, then the following condtions are equivalent:*

*(i)* $(Z^{(n)})_{n\in\mathbb{N}^*}$ *is UT,*

*(ii) for each* $t>0$ *and each* $\epsilon>0$*, there exists* $a,M>0$ *such that*

$$\sup_{n\geq 1}\mathbf{P}^{(n)}\left(\int_0^t\int_{\mathbb{R}}|x|\mathbf{1}_{\{|x|>a\}}\nu^{(n)}(ds,dx)>M\right)<\epsilon.$$

*Proof.* From Lemma 3.1. in [Jakubowski et al., 1989] we know that, under the assumption $Z^{(n)}\xrightarrow{d}Z$ as $n\to\infty$, (i) is equivalent to asking that for each $t>0$ and each $\epsilon>0$, there exists $M>0$ such that

$$\sup_{n\geq 1}\mathbf{P}^{(n)}(V(B^{a,n})_t>M)<\epsilon,$$

where $V(.)$ is the total first order variation of a process and $B^{a,n}$ is the first semimartingale characteristic of $Z^{(n)}$ (for the truncation function $h(x)=x\mathbf{1}_{\{|x|>a\}}$).

Let's compute $V(B^{a,n})$ in this case. For $a>0$ and $n\in\mathbb{N}^*$, define $\tilde{Z}_t^{n,a}=Z_t^{(n)}-\sum_{0<s\leq t}\Delta Z_s\mathbf{1}_{\{|\Delta Z_s|>a\}}$ and $B_t^{a,n}=\int_0^t\int_{\mathbb{R}}x\mathbf{1}_{\{|x|>a\}}\nu^{(n)}(ds,dx)$. We have,

$$\begin{aligned}\tilde{Z}_t^{n,a}&=\tilde{Z}_t^{n,a}+B_t^{a,n}-B_t^{a,n}\\&=Z_t^{(n)}-\int_0^t\int_{\mathbb{R}}x\mathbf{1}_{\{|x|>a\}}(\mu^{(n)}(ds,dx)-\nu^{(n)}(ds,dx))-B_t^{a,n},\end{aligned}$$

where $\mu^{(n)}$ is the jump measure of $Z^{(n)}$. Thus, since the two first terms on the r.h.s. of the last line above are local martingales, their difference is a local martingale with bounded jumps and thus the first semimartingale characteristic of $Z^{(n)}$ is $B_t^{a,n}$. So,

$$V(B^{a,n})_t=\int_0^t\int_{\mathbb{R}}|x|\mathbf{1}_{\{|x|>a\}}\nu^{(n)}(ds,dx)$$

and this finishes the proof. □

We are now ready for the proof of Theorem 2.2.1.

*Proof of Theorem 2.2.1.* To be able to apply Proposition 2.2.9, we need show that $(e^{R^{(n)}},X^{(n)})_{n\in\mathbb{N}^*}$ converges in law as $n\to\infty$ and that $(X^{(n)})_{n\in\mathbb{N}^*}$ is UT.

First, note that by definition of $\gamma_k^{(n)}$, we have

$$R_t^{(n)} = \sum_{i=1}^{[nt]} \gamma_k^{(n)} = \mu_\rho \frac{[nt]}{n} + \sum_{i=1}^{[nt]} \frac{\ln(\rho_i) - \mu_\rho}{c_\beta n^{1/\beta}}. \tag{2.8}$$

But $[nt]/n \to t$ as $n \to \infty$. By the stable functional convergence theorem (see e.g. Theorem 2.4.10 p.95 in [Embrechts et al., 1997]), the sum in the r.h.s. of the equation above converges weakly to a stable Lévy process $(L_t^\beta)_{t\geq 0}$ with $L_1^\beta \stackrel{d}{=} K_\beta$. Thus, we obtain

$$(e^{-R_t^{(n)}})_{t\geq 0} = \left(\exp\left(-\sum_{i=1}^{[nt]} \gamma_k^{(n)}\right)\right)_{t\geq 0} \stackrel{d}{\to} \left(e^{-\mu_\rho t - L_t^\beta}\right)_{t\geq 0}.$$

Similarly, by the definition of $\xi_i^{(n)}$, we have

$$X_t^{(n)} = \sum_{i=1}^{[nt]} \frac{\mu_\xi}{n} + \sum_{i=1}^{[nt]} \frac{\xi_i - \mu_\xi}{c_\alpha n^{1/\alpha}} = \mu_\xi A_t^{(n)} + N_t^{(n)}, \text{ for all } t \geq 0. \tag{2.9}$$

Applying the stable functional convergence theorem again, we obtain $(N_t^{(n)})_{t\geq 0} \stackrel{d}{\to} (L_t^\alpha)_{t\geq 0}$, as $n \to \infty$, where $L^\alpha$ is a stable Lévy motion, with $L_1^\alpha \stackrel{d}{=} K_\alpha$, which is independent of $(L_t^\beta)_{t\geq 0}$ since the sequences $(\xi_k)_{k\in\mathbb{N}^*}$ and $(\rho_k)_{k\in\mathbb{N}^*}$ are independent. Using the independence, we also have the convergence of the couple $(e^{R^{(n)}}, X^{(n)})$, as $n \to \infty$.

To prove that $(X^{(n)})_{n\in\mathbb{N}^*}$ is UT, it is enough to prove that $(A^{(n)})_{n\in\mathbb{N}^*}$ and $(N^{(n)})_{n\in\mathbb{N}^*}$ are both UT. Note that $A^{(n)}$ is a process of locally bounded variation for each $n \geq 1$ with $V(A^{(n)}) = A^{(n)}$. Since $A_t^{(n)} \leq t$, for all $n \in \mathbb{N}^*$, we have

$$\sup_{n\geq 1} \mathbf{P}(A_t^{(n)} > M) \leq \mathbf{P}(t > M),$$

for all $M > 0$ and thus, by Lemma 2.2.10, the sequence $(A^{(n)})_{n\in\mathbb{N}^*}$ is UT.

Now, note that, when $t > s$ and $[nt] \geq [ns] + 1$, using the i.i.d. property of $(\xi_k)_{k\in\mathbb{N}^*}$ we obtain

$$\mathbf{E}(N_t^{(n)} - N_s^{(n)} | \mathcal{F}_s) = \sum_{i=[ns]+1}^{[nt]} \mathbf{E}\left(\frac{\xi_i - \mu_\xi}{c_\alpha n^{1/\alpha}}\right) = 0.$$

When $t > s$ and $[nt] < [ns]+1$, $N_t^{(n)} - N_s^{(n)} = 0$, and thus $\mathbf{E}(N_t^{(n)} - N_s^{(n)} | \mathcal{F}_s) = 0$. This shows that $N^{(n)}$ is a local martingale for each $n \in \mathbb{N}^*$. Then, denoting by $\nu^{(n)}$ the compensator of the jump measure of $N^{(n)}$ (which is deterministic since $N^{(n)}$ is also a semimartingale with independent increments), we set

$$s_n = \int_0^t \int_{\mathbb{R}} |x| \mathbf{1}_{\{|x|>1\}} \nu^{(n)}(ds, dx),$$

for each $n \in \mathbb{N}^*$, and we will show that the (deterministic) sequence $(s_n)_{n \in \mathbb{N}^*}$ converges (and thus is bounded).

First, we have

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}} |x| \mathbf{1}_{\{|x|>1\}} \nu^{(n)}(ds, dx) &= \mathbf{E}\left( \sum_{0<s\le t} |\Delta N_s^{(n)}| \mathbf{1}_{\{|\Delta N^{(n)}| \ge 1\}} \right) \\
&= \sum_{i=1}^{[nt]} \mathbf{E}\left( \left| \frac{\xi_i - \mu_\xi}{c_\alpha n^{1/\alpha}} \right| \mathbf{1}_{\left\{ \left| \frac{\xi_i - \mu_\xi}{c_\alpha n^{1/\alpha}} \right| \ge 1 \right\}} \right) \\
&= \frac{[nt]}{c_\alpha n^{1/\alpha}} \mathbf{E}\left( |\xi_1 - \mu_\xi| \, \mathbf{1}_{\{|\xi_1 - \mu_\xi| \ge c_\alpha n^{1/\alpha}\}} \right).
\end{aligned}$$

To compute the expectation on the r.h.s., note that for any non-negative random variable $Z$ and constant $a \ge 0$ we have

$$\begin{aligned}
\mathbf{E}(Z \mathbf{1}_{\{Z \ge a\}}) = \mathbf{E}\left( \int_0^Z \mathbf{1}_{\{Z \ge a\}} dx \right) &= \mathbf{E}\left( \int_0^\infty \mathbf{1}_{\{Z \ge x \vee a\}} dx \right) \\
&= \int_0^\infty \mathbf{P}(Z \ge x \vee a) dx \\
&= a \mathbf{P}(Z \ge a) + \int_a^\infty \mathbf{P}(Z \ge x) dx.
\end{aligned}$$

Thus,

$$\begin{aligned}
s_n &= \frac{[nt]}{c_\alpha n^{1/\alpha}} \mathbf{E}\left( (\xi_1 - \mu_\xi) \, \mathbf{1}_{\{(\xi_1 - \mu_\xi) \ge c_\alpha n^{1/\alpha}\}} \right) \\
&\quad + \frac{[nt]}{c_\alpha n^{1/\alpha}} \mathbf{E}\left( -(\xi_1 - \mu_\xi) \, \mathbf{1}_{\{-(\xi_1 - \mu_\xi) \ge c_\alpha n^{1/\alpha}\}} \right) \\
&= [nt] \mathbf{P}\left( \xi_1 \ge \mu_\xi + c_\alpha n^{1/\alpha} \right) + \frac{[nt]}{c_\alpha n^{1/\alpha}} \int_{c_\alpha n^{1/\alpha}}^\infty \mathbf{P}\left( \xi_1 \ge \mu_\xi + x \right) dx \\
&\quad + [nt] \mathbf{P}\left( \xi_1 \le \mu_\xi - c_\alpha n^{1/\alpha} \right) + \frac{[nt]}{c_\alpha n^{1/\alpha}} \int_{c_\alpha n^{1/\alpha}}^\infty \mathbf{P}\left( \xi_1 \le \mu_\xi - x \right) dx.
\end{aligned}$$

Using the fact that $\xi_1$ satisfies $(\mathbf{H}^\alpha)$, we see that $\mathbf{P}\left(\xi_1 \leq \mu_\xi - c_\alpha x^{1/\alpha}\right) \sim k_1^{\xi_1} c_\alpha^{-\alpha} x^{-1}$ and $\mathbf{P}\left(\xi_1 \geq \mu_\xi + c_\alpha x^{1/\alpha}\right) \sim k_2^{\xi_1} c_\alpha^{-\alpha} x^{-1}$, as $x \to \infty$. So,

$$\begin{aligned}
\lim_{n\to\infty} s_n &= \lim_{n\to\infty} \frac{k_1^{\xi_1}}{c_\alpha^\alpha}\frac{[nt]}{n} - \lim_{n\to\infty} \frac{[nt]}{c_\alpha n^{1/\alpha}}\frac{k_1^{\xi_1} c_\alpha^{1-\alpha} n^{(1-\alpha)/\alpha}}{1-\alpha} \\
&+ \lim_{n\to\infty} \frac{k_2^{\xi_1}}{c_\alpha^\alpha}\frac{[nt]}{n} - \lim_{n\to\infty} \frac{[nt]}{c_\alpha n^{1/\alpha}}\frac{k_2^{\xi_1} c_\alpha^{1-\alpha} n^{(1-\alpha)/\alpha}}{1-\alpha} \\
&= \frac{k_1^{\xi_1}+k_2^{\xi_1}}{c_\alpha^\alpha}\frac{\alpha}{\alpha-1} t.
\end{aligned} \tag{2.10}$$

Thus, the sequence is bounded and taking $M > 0$ large enough, we find $\sup_{n\geq 1} \mathbf{P}(s_n > M) < \epsilon$, for each $\epsilon > 0$, and, by Lemma 2.2.11, we have then shown that the sequence $(N^{(n)})_{n\in\mathbb{N}^*}$ is UT.

To conclude we obtain, using Proposition 2.2.9 and the continuous mapping theorem with $h(x_1, x_2, x_3) = (x_3 + y)/x_2$, $(\theta_t^{(n)})_{t\geq 0} \xrightarrow{d} (Y_t)_{t\geq 0}$ where $Y = (Y_t)_{t\geq 0}$ is given by (2.2) with $R_t = \mu_\rho t + L_t^\beta$, $X_t = \mu_\xi t + L_t^\alpha$, for all $t \geq 0$.

In this case, we have $[R, X]_t = 0$, for all $t \geq 0$, (see Proposition 1.3.3) and thus, using Itô's lemma and Proposition 1.2.12, we obtain the stochastic differential equation (2.5). □

*Proof of Theorem 2.2.3.* We start by proving that $\mathbf{P}(\inf_{0\leq t\leq T} Y_t = 0) = 0$. First, note that

$$\left\{\inf_{0\leq t\leq T} Y_t = 0\right\} = \left\{\sup_{0\leq t\leq T}\left(-\int_{0+}^t e^{-R_{s-}} dX_s\right) = y\right\}.$$

Using the independence of the processes, we then obtain

$$\mathbf{P}\left(\inf_{0\leq t\leq T} Y_t = 0\right) = \int_{\mathbb{D}} \mathbf{P}\left(\sup_{0\leq t\leq T}\left(-\int_{0+}^t g(s-) dX_s\right) = y\right) \mathbf{P}_{e^{-R}}(dg)$$

where $\mathbb{D}$ is the space of càdlàg functions and $\mathbf{P}_{e^{-R}}$ is the law of the process $(e^{-R_t})_{t\geq 0}$. Denote $S(g)_t = -\int_{0+}^t g(s-) dX_s$, for all $t \geq 0$.

Let $(t_i)_{i\in\mathbb{N}^*}$ be an enumerating sequence of $[0, T] \cap \mathbb{Q}$. Since $S(g) = (S(g)_t)_{t\geq 0}$ is a process with independent increments, $S(g)$ has, for each fixed time $t_i > 0$, the same law as a Lévy process $L = (L_t)_{t\geq 0}$ defined by the characteristic triplet $(a_L, \sigma_L^2, \nu_L)$ with

$$a_L = \frac{\mu_\xi}{t_i}\int_0^{t_i} g(s-)ds, \;\; \sigma_L^2 = \frac{\sigma_\xi^2}{t_i}\int_0^{t_i} g^2(s-)ds$$

and

$$\nu_L(dx) = \frac{\nu_\xi(dx)}{t_i}\int_0^{t_i} g(s-)ds,$$

where $(a_\xi, \sigma_\xi^2, \nu_\xi)$ is the characteristic triplet of $X$, see Theorem 4.25 p.110 in [Jacod and Shiryaev, 2003]. Then, it is well known that $L_{t_i}$ admits a density if $\sigma_L^2 > 0$ or $\nu_L(\mathbb{R}) = \infty$, see e.g. Proposition 3.12 p.90 in [Cont and Tankov, 2004]. But, when $\xi_1$ satisfies $(\mathbf{H^2})$, we have $\sigma_\xi^2 > 0$ and $\sigma_L^2 > 0$. When $\xi_1$ satisfies $(\mathbf{H}^\alpha)$, we have $\nu_\xi(\mathbb{R}) = \infty$ and $\nu_L(\mathbb{R}) = \infty$. Thus, in both cases, $L_{t_i}$ admits a density and we have $\mathbf{P}(S(g)_{t_i} = y) = \mathbf{P}(L_{t_i} = y) = 0$.

Since $(S(g)_t)_{t\geq 0}$ is càdlàg we have

$$\sup_{0\leq t\leq T} S(g)_t = \sup_{t\in[0,T]\cap\mathbb{Q}} S(g)_t,$$

and, since a càdlàg process reaches its supremum almost surely,

$$\begin{aligned}\mathbf{P}\left(\sup_{0\leq t\leq T} S(g)_t = y\right) &= \mathbf{P}\left(\sup_{t\in[0,T]\cap\mathbb{Q}} S(g)_t = y\right) \leq \mathbf{P}\left(\bigcup_{i\in\mathbb{N}}\{S_{t_i}(g) = y\}\right)\\ &= \lim_{N\to\infty}\mathbf{P}\left(\bigcup_{i=1}^{N}\{S_{t_i}(g) = y\}\right) \leq \lim_{N\to\infty}\sum_{i=1}^{N}\mathbf{P}(S_{t_i}(g) = y) = 0.\end{aligned}$$

Thus, $\mathbf{P}(\inf_{0\leq t\leq T} Y_t = 0) = 0$.

Next, note that we have

$$\left\{\inf_{0\leq t\leq T} Y_t < 0\right\} \subseteq \{\tau(y) \leq T\} \subseteq \left\{\inf_{0\leq t\leq T} Y_t \leq 0\right\}$$

and

$$\left\{\inf_{0\leq t\leq T} \theta_t^{(n)} < 0\right\} \subseteq \{\tau^n(y) \leq T\} \subseteq \left\{\inf_{0\leq t\leq T} \theta_t^{(n)} \leq 0\right\}.$$

Since $\theta^{(n)} \xrightarrow{d} Y$ by Theorem 2.2.1, we obtain from the continuous mapping theorem that $\inf_{0\leq t\leq T}\theta_t^{(n)} \xrightarrow{d} \inf_{0\leq t\leq T} Y_t$, for all $T \geq 0$, since the supremum (and also the infimum) up to a fixed time are continuous for the Skorokhod topology (see e.g. Proposition 2.4, p.339, in [Jacod and Shiryaev, 2003]). So, by the portmanteau theorem,

$$\begin{aligned}\limsup_{n\to\infty} \mathbf{P}(\tau^n(y) \leq T) &\leq \limsup_{n\to\infty} \mathbf{P}\left(\inf_{0\leq t\leq T} \theta_t^{(n)} \leq 0\right) \\ &\leq \mathbf{P}\left(\inf_{0\leq t\leq T} Y_t \leq 0\right) = \mathbf{P}\left(\inf_{0\leq t\leq T} Y_t < 0\right) \\ &\leq \mathbf{P}(\tau(y) \leq T),\end{aligned}$$

and

$$\begin{aligned}\liminf_{n\to\infty} \mathbf{P}(\tau^n(y) \leq T) &\geq \liminf_{n\to\infty} \mathbf{P}\left(\inf_{0\leq t\leq T} \theta_t^{(n)} < 0\right) \\ &\geq \mathbf{P}\left(\inf_{0\leq t\leq T} Y_t < 0\right) = \mathbf{P}\left(\inf_{0\leq t\leq T} Y_t \leq 0\right) \\ &\geq \mathbf{P}(\tau(y) \leq T).\end{aligned}$$

□

We will now turn to the application of the results of this section to the approximation of certain functionals. We mention here, that we only prove the convergence of the approximation and leave the question of rate of convergence to future research.

## 2.3 Approximation of the Ultimate Ruin Probability

In this section, we deduce an approximation of the ultimate ruin probability of the discrete-time process. In order to be able to go further (and to obtain practical expressions for the ultimate ruin probability and the moments of the limiting process), we will from now on restrict ourselves to the $(\mathbf{H^2})$ case which we summarize in the following assumption.

**Assumption** $(\mathbf{H'})$**.** We assume that $\xi_1$ and $\ln(\rho_1)$ both satisfy $(\mathbf{H^2})$. So $Y$ is given by (2.2) with $X_t = \mu_\xi t + \sigma_\xi \tilde{W}_t$ and $R_t = \mu_\rho t + \sigma_\rho W_t$ or, equivalently, is given by the solution of (2.5) with the same $X$ and $\tilde{R}_t = \kappa_\rho t + \sigma_\rho W_t$ and $\kappa_\rho = \mu_\rho + \sigma_\rho^2/2$.

We have seen that, when $\xi_1$ and $\ln(\rho_1)$ both satisfy $(\mathbf{H^2})$, we have

$$\lim_{n\to\infty} \mathbf{P}(\tau^n(y) \leq T) = \mathbf{P}(\tau(y) \leq T),$$

for all $T \geq 0$. We would like to replace the finite-time ruin probability with the ultimate ruin probability $\mathbf{P}(\tau(y) < \infty)$ since for the latter, an explicit expression exists for the limiting process. However, the following classic example (see e.g. [Grandell, 1977]) shows that the ultimate ruin probability may fail to converge even if the finite-time ruin probability does. In fact, take $(Z^{(n)})_{t\geq 0}$ to be the deterministic process defined by

$$Z_t^{(n)} = \begin{cases} 0 \text{ if } t < n, \\ -1 \text{ if } t \geq n. \end{cases}$$

Then, we have $Z^{(n)} \to Z$, as $n \to \infty$, where $Z_t = 0$, for all $t \geq 0$, and we have also convergence of the finite-time ruin probability, since, as $n \to \infty$, $\inf_{0\leq t\leq T} Z_t^{(n)} \to 0$, for all $T > 0$. But $\inf_{0\leq t<\infty} Z_t^{(n)} = -1$, for all $n \in \mathbb{N}^*$, and so the ultimate ruin probability fails to converge.

In general, proving the convergence of the ultimate ruin probability is a hard problem and depends on the particular model (see [Grandell, 1977] for another discussion). Still, we now give a sufficient condition for this convergence.

**Theorem 2.3.1.** *Assume that* $(\mathbf{H}')$ *holds. When* $\mu_\rho \leq 0$*, we have*

$$\lim_{n\to\infty} \mathbf{P}(\tau^n(y) < \infty) = 1.$$

*When* $\mu_\rho > 0$*, we assume additionally that there exists* $C < 1$ *and* $n_0 \in \mathbb{N}^*$ *such that*

$$\sup_{n\geq n_0} \mathbf{E}\left(e^{-2\gamma_1^{(n)}}\right)^n = \sup_{n\geq n_0} \mathbf{E}\left((\rho_1^{(n)})^{-2}\right)^n \leq C. \tag{2.11}$$

*Then,*

$$\lim_{n\to\infty} \mathbf{P}(\tau^n(y) < \infty) = \mathbf{P}(\tau(y) < \infty) = \frac{H(-y)}{H(0)}$$

*where, for* $x \leq 0$*,*

$$H(x) = \int_{-\infty}^{x} (\sigma_\xi^2 + \sigma_\rho^2 z^2)^{-(1/2+\mu_\rho/\sigma_\rho^2)} \exp\left(\frac{2\mu_\xi}{\sigma_\xi\sigma_\rho} \arctan\left(\frac{\sigma_\rho}{\sigma_\xi} z\right)\right) dz.$$

Before turning to the proof of the theorem, we give two examples to illustrate Condition (2.11).

**Example 2.3.2** (Approximation of the ruin probability with normal log-returns)**.** *Take* $\xi_1$ *to be any random variable satisfying* $(\mathbf{H^2})$ *and* $\ln(\rho_1) \sim \mathcal{N}(\mu_\rho, \sigma_\rho^2)$*, with* $\mu_\rho > 0$*, then*

$$\mathbf{E}\left(e^{-2\gamma_1^{(n)}}\right)^n = e^{-2(\mu_\rho - \sigma_\rho^2)},$$

*for all* $n \in \mathbb{N}^*$*, so* $n_0 = 1$ *and the condition* $C < 1$ *is equivalent to* $\mu_\rho > \sigma_\rho^2$*.*

**Example 2.3.3** (Approximation of the ruin probability with NIG log-returns)**.** More generally, take $\xi_1$ to be any random variable satisfying $(\mathbf{H^2})$ and $\ln(\rho_1)$ to be a normal inverse gaussian NIG$(\alpha, \beta, \delta, \mu)$ random variable with $0 \leq |\beta| < \alpha$, $\delta > 0$ and $\mu \in \mathbb{R}$ (recall Example 2.2.6 for the definition). We remark that when $n$ is large enough, we can use Taylor's formula, to obtain

$$\sqrt{\alpha^2 - \left(\beta - \frac{2}{\sqrt{n}}\right)^2} = \gamma + \frac{2}{\sqrt{n}}\frac{\beta}{\gamma} - \frac{2}{n}\frac{\alpha^2}{[\alpha^2 - (\beta - x_n)^2]^{3/2}}, \tag{2.12}$$

for some $x_n \in [0, 2/\sqrt{n}]$. Since the mean is given by $\mu_\rho = \mu + \delta\beta/\gamma$, we obtain using (2.12)

$$\begin{aligned}\lim_{n\to\infty} \mathbf{E}\left(e^{-2\gamma_1^{(n)}}\right)^n &= \exp\left(-2\mu_\rho + \frac{2\delta\alpha^2}{\gamma^3}\right)\\ &= \exp\left(-2\left(\mu + \frac{\delta\beta\gamma^2 - \delta\alpha^2}{\gamma^3}\right)\right)\end{aligned}$$

Thus, when

$$\mu + \frac{\delta\beta\gamma^2 - \delta\alpha^2}{\gamma^3} > 0,$$

this limit is strictly smaller than 1 and we can find $n_0 \in \mathbb{N}^*$ and $C < 1$ such that (2.11) is satisfied. Taking $\beta = 0$ and $\sigma^2 = \delta/\alpha$ we retrieve the condition for normal returns given in Example 2.3.2.

*Proof of Theorem 2.3.1.* We have, for all $n \in \mathbb{N}^*$ and $T > 0$,

$$\mathbf{P}(\tau^n(y) < \infty) \geq \mathbf{P}(\tau^n(y) \leq T)$$

and, by Theorem 2.2.3,

$$\liminf_{n\to\infty} \mathbf{P}(\tau^n(y) < \infty) \geq \mathbf{P}(\tau(y) \leq T).$$

So, letting $T \to \infty$,

$$\liminf_{n\to\infty} \mathbf{P}(\tau^n(y) < \infty) \geq \mathbf{P}(\tau(y) < \infty).$$

Now if $\mathbf{P}(\tau(y) < \infty) = 1$, which is equivalent to $\mu_\xi \leq 0$ by [Paulsen, 1998], there is nothing else to prove. So we assume that $\mathbf{P}(\tau(y) < \infty) < 1$, or $\mu_\xi > 0$, and we will prove that

$$\limsup_{n\to\infty} \mathbf{P}(\tau^n(y) < \infty) \leq \mathbf{P}(\tau(y) < \infty),$$

under the additional condition (2.11).

Fix $y > \epsilon > 0$, $T > 0$ and, when $\tau^n(y) > T$, denote by $K_{\epsilon,T}^{(n)}$ the event

$$K_{\epsilon,T}^{(n)} = \left\{ \left| \int_{T+}^{\tau^n(y)} e^{-R_{s-}^{(n)}} dX_s^{(n)} \right| < \epsilon \right\}.$$

We have,

$$\begin{aligned}\{\tau^n(y) < \infty\} = \{\tau^n(y) \leq T\} &\cup \{\tau^n(y) \in (T,\infty), K_{\epsilon,T}^{(n)}\} \\ &\cup \{\tau^n(y) \in (T,\infty), (K_{\epsilon,T}^{(n)})^{\complement}\}.\end{aligned}$$

But, on the event $\{\tau^n(y) \in (T,\infty), K_{\epsilon,T}^{(n)}\}$,

$$\int_{0+}^{T} e^{-R_{s-}^{(n)}} dX_s^{(n)} + \int_{T+}^{\tau^n(y)} e^{-R_{s-}^{(n)}} dX_s^{(n)} < -y$$

which implies

$$\int_{0+}^{T} e^{-R_{s-}^{(n)}} dX_s^{(n)} < -y + \epsilon,$$

or equivalently that $\tau^n(y-\epsilon) \leq T$, by (2.7). Thus,

$$\{\tau^n(y) \leq T\} \cup \{\tau^n(y) \in (T,\infty), K_{\epsilon,T}^{(n)}\} \subseteq \{\tau^n(y-\epsilon) \leq T\}.$$

Then, we have $\{\tau^n(y) \in (T,\infty), (K^{(n)}_{\epsilon,T})^{\complement}\} \subseteq (K^{(n)}_{\epsilon,T})^{\complement}$ and thus

$$\limsup_{n\to\infty} \mathbf{P}(\tau^n(y) < \infty) \leq \mathbf{P}(\tau(y-\epsilon) \leq T) + \limsup_{n\to\infty} \mathbf{P}\left((K^{(n)}_{\epsilon,T})^{\complement}\right).$$

So, we need to show that

$$\lim_{T\to\infty} \limsup_{n\to\infty} \mathbf{P}\left((K^{(n)}_{\epsilon,T})^{\complement}\right) = 0.$$

Using the decomposition (2.9), we obtain

$$(K^{(n)}_{\epsilon,T})^{\complement} \subseteq \left\{\left|\int_{T+}^{\tau^n(y)} e^{-R^{(n)}_{s-}} dA^{(n)}_s\right| \geq \frac{\epsilon}{2}\right\} \cup \left\{\left|\int_{T+}^{\tau^n(y)} e^{-R^{(n)}_{s-}} dN^{(n)}_s\right| \geq \frac{\epsilon}{2}\right\}$$

Denote by $E^{(n)}_{1,T}$ and $E^{(n)}_{2,T}$ the sets on the r.h.s. of the above equation.

When $n \geq n_0$, we obtain, recalling the explicit form of the integral and using Markov's inequality,

$$\begin{aligned}
\mathbf{P}(E^{(n)}_{1,T}) &\leq \frac{2|\mu_\xi|}{n\epsilon}\mathbf{E}\left(\sum_{i=[nT]+1}^{[n\tau^n(y)]+1} e^{-\sum_{j=1}^{i}\gamma_j^{(n)}}\right) \\
&\leq \frac{2|\mu_\xi|}{n\epsilon}\mathbf{E}\left(\sum_{i=[nT]+1}^{\infty}\prod_{j=1}^{i} e^{-\gamma_j^{(n)}}\right) = \frac{2|\mu_\xi|}{n\epsilon}\sum_{i=[nT]+1}^{\infty}\mathbf{E}\left(e^{-\gamma_1^{(n)}}\right)^i \\
&= \frac{2|\mu_\xi|}{n\epsilon}\mathbf{E}\left(e^{-\gamma_1^{(n)}}\right)^{[nT]}\sum_{j=1}^{\infty}\mathbf{E}\left(e^{-\gamma_1^{(n)}}\right)^j.
\end{aligned}$$

But, since $\mathbf{E}(e^{-\gamma_1^{(n)}}) \leq \mathbf{E}(e^{-2\gamma_1^{(n)}})^{1/2} \leq C^{1/(2n)} < 1$, we have

$$\mathbf{P}(E^{(n)}_{1,T}) \leq \frac{2|\mu_\xi|}{\epsilon}\frac{C^{1/(2n)}}{n(1-C^{1/(2n)})}C^T.$$

Moreover it is easy to see that $C^{-1/(2n)}(n(1-C^{-1/(2n)}))^{-1} \to -2/\ln(C)$ as $n \to \infty$, and so $\lim_{T\to\infty}\limsup_{n\to\infty}\mathbf{P}(E^{(n)}_{1,T}) = 0$.

On the other hand, using the Chebyshev and Burkholder-Davis-Gundy inequalities, we obtain

$$
\begin{aligned}
\mathbf{P}(E_{2,T}^{(n)}) &\leq \frac{4}{\epsilon^2}\mathbf{E}\left(\left|\int_{T+}^{\tau^n(y)} e^{-R_{s-}^{(n)}}\,dN_s^{(n)}\right|^2\right) \\
&\leq \frac{4}{\epsilon^2}\mathbf{E}\left(\sup_{T<t<\infty}\left|\int_{T+}^{t} e^{-R_{s-}^{(n)}}\,dN_s^{(n)}\right|^2\right) \\
&\leq \frac{4K}{\epsilon^2}\mathbf{E}\left(\int_{T+}^{\infty} e^{-2R_{s-}^{(n)}}\,d[N^{(n)},N^{(n)}]_s\right),
\end{aligned}
$$

where $K$ is a constant. But,

$$
[N^{(n)},N^{(n)}]_t = \sum_{0<s\leq t}(\Delta N_s^{(n)})^2 = \sum_{i=1}^{[nt]}\left(\frac{\xi_i-\mu_\xi}{\sqrt{n}}\right)^2.
$$

Thus, writing the stochastic integral explicitly and using the same computation as before, we obtain

$$
\begin{aligned}
\mathbf{P}(E_{2,T}^{(n)}) &\leq \frac{4K}{\epsilon^2}\mathbf{E}\left(\sum_{i=[nT]+1}^{\infty}\left(\frac{\xi_i-\mu_\xi}{\sqrt{n}}\right)^2 e^{-2\sum_{j=1}^{i}\gamma_j^{(n)}}\right) \\
&= \frac{4K\sigma_\xi^2}{\epsilon^2 n}\sum_{i=[nT]+1}^{\infty}\mathbf{E}(e^{-2\gamma_1^{(n)}})^i \leq \frac{4K}{\epsilon^2}C^T\sigma_\xi^2\frac{C^{1/n}}{n(1-C^{1/n})}.
\end{aligned}
$$

Again, using the fact that the expression on the r.h.s. above converges, when $n\to\infty$, we find that

$$
\lim_{T\to\infty}\limsup_{n\to\infty}\mathbf{P}\left(E_{2,T}^{(n)}\right) = 0
$$

and

$$
\limsup_{n\to\infty}\mathbf{P}(\tau^n(y)<\infty) \leq \mathbf{P}(\tau(y-\epsilon)<\infty).
$$

So, letting $\epsilon\to 0$ and using the continuity of $y\mapsto\mathbf{P}(\tau(y)<\infty)$, we obtain

$$
\limsup_{n\to\infty}\mathbf{P}(\tau^n(y)<\infty) \leq \mathbf{P}(\tau(y)<\infty).
$$

The explicit expression for the ultimate ruin probability of the limiting process is given in [Paulsen and Gjessing, 1997]. $\square$

## 2.4 Approximation of the Moments

In this section, we obtain a recursive formula for the moments of the limiting process $Y$ at a fixed time which, for simplicity, we choose to be $T = 1$ and prove the convergence of the moments of $\theta_1^{(n)}$ to the moments of $Y_1$. This gives a way to approximate the moments of $\theta_1^{(n)}$.

**Proposition 2.4.1.** *Assume that the limiting process $Y = (Y_t)_{t\geq 0}$ is given by (2.2) with $X_t = \mu_\xi t + \sigma_\xi \tilde{W}_t$ and $R_t = \mu_\rho t + \sigma_\rho W_t$, for all $t \geq 0$. We have, for all $p \in \mathbb{N}$,*

$$\mathbf{E}\left(\sup_{0\leq t\leq 1} |Y_t|^p\right) < \infty. \tag{2.13}$$

*Moreover, letting $m_p(t) = \mathbf{E}[(Y_t)^p]$, for each $0 \leq t \leq 1$ and $p \in \mathbb{N}$, we have the following recursive formula: $m_0(t) = 1$,*

$$m_1(t) = \begin{cases} ye^{\kappa_\rho t} + \frac{\mu_\xi}{\kappa_\rho}(e^{\kappa_\rho t} - 1) \text{ when } \kappa_\rho \neq 0, \\ y + \mu_\xi t \text{ when } \kappa_\rho = 0, \end{cases} \tag{2.14}$$

*with $\kappa_\rho = \mu_\rho + \sigma_\rho^2/2$ and, for each $p \geq 2$,*

$$m_p(t) = y^p e^{a_p t} + \int_0^t e^{a_p(t-s)} \left(b_p m_{p-1}(s) + c_p m_{p-2}(s)\right) ds, \tag{2.15}$$

*with $a_p = p\mu_\rho + p^2\sigma_\rho^2/2$, $b_p = p\mu_\xi$ and $c_p = p(p-1)\sigma_\xi^2/2$.*

*Proof.* The existence of the moments (2.13) follows, for $p \geq 2$, from the general existence result for the strong solutions of SDEs, see e.g. Corollary 2.2.1 p.119 in [Nualart, 2006] and, for $p = 1$, from Cauchy-Schwarz's inequality.

Set $m_p(t) = \mathbf{E}[(Y_t)^p]$, for all $0 \leq t \leq 1$ and $p \in \mathbb{N}^*$. Suppose that $p \geq 2$. For $r \in \mathbb{N}^*$, define the stopping times

$$\theta_r = \inf\{t > 0 : |Y_t| > r\}$$

with $\inf \emptyset = +\infty$. Then, applying Itô's lemma and using $\langle Y, Y\rangle_t = \sigma_\xi^2 t +$

$\sigma_\rho^2 \int_0^t Y_s^2 ds$, yields

$$\begin{aligned}(Y_{t\wedge\theta_r})^p = y^p &+ p\mu_\xi \int_0^{t\wedge\theta_r} (Y_s)^{p-1} ds + p\sigma_\xi \int_0^{t\wedge\theta_r} (Y_s)^{p-1} d\tilde{W}_s \\ &+ p\kappa_\rho \int_0^{t\wedge\theta_r} (Y_s)^p ds + p\sigma_\rho \int_0^{t\wedge\theta_r} (Y_s)^p dWs \\ &+ \frac{p(p-1)}{2}\sigma_\xi^2 \int_0^{t\wedge\theta_r} (Y_s)^{p-2} ds \\ &+ \frac{p(p-1)}{2}\sigma_\rho^2 \int_0^{t\wedge\theta_r} (Y_s)^p ds.\end{aligned}$$

Thus, using Fubini's theorem and the fact that the stochastic integrals are martingales, we obtain

$$\begin{aligned}\mathbf{E}[(Y_{t\wedge\theta_r})^p] = y^p &+ p\mu_\xi \int_0^{t\wedge\theta_r} \mathbf{E}[(Y_s)^{p-1}] ds + p\kappa_\rho \int_0^{t\wedge\theta_r} \mathbf{E}[(Y_s)^p] ds \\ &+ \frac{p(p-1)}{2}\sigma_\xi^2 \int_0^{t\wedge\theta_r} \mathbf{E}[(Y_s)^{p-2}] ds \\ &+ \frac{p(p-1)}{2}\sigma_\rho^2 \int_0^{t\wedge\theta_r} \mathbf{E}[(Y_s)^p] ds.\end{aligned}$$

Now we can take the limit as $r \to \infty$, and use (2.13) to pass it inside the expectation of the l.h.s. of the above equation. Differentiating w.r.t. $t$, we then obtain the following ODE

$$\begin{aligned}\frac{d}{dt}\mathbf{E}[(Y_t)^p] = &\left(p\kappa_\rho + \frac{p(p-1)}{2}\sigma_\rho^2\right)\mathbf{E}[(Y_t)^p] + p\mu_\xi \mathbf{E}[(Y_t)^{p-1}] \\ &+ \frac{p(p-1)}{2}\sigma_\xi^2 \mathbf{E}[(Y_t)^{p-2}],\end{aligned}$$

and $\mathbf{E}[(Y_0)^p] = y^p$. This is an inhomogeneous linear equation of the first order which can be solved explicitly to obtain (2.15).

For $p = 1$, using the same technique as above, we obtain

$$\mathbf{E}(Y_t) = y + \mu_\xi t + \kappa_\rho \int_0^t \mathbf{E}(Y_s) ds.$$

If $\kappa_\rho = 0$, there is nothing to prove. If $\kappa_\rho \neq 0$, we obtain by differentiating w.r.t. $t$,

$$\frac{d}{dt}\mathbf{E}(Y_t) = \mu_\xi + \kappa_\rho \mathbf{E}(Y_t),$$

with $\mathbf{E}(Y_0) = y$ and this can be solved to obtain (2.14). □

We now state the approximation result.

**Theorem 2.4.2.** *Assume that* $(\mathbf{H}')$ *holds. Assume that* $\mathbf{E}(|\xi_1|^q) < \infty$*, and that*

$$\sup_{n\in\mathbb{N}^*} \mathbf{E}\left(e^{q\gamma_1^{(n)}}\right)^n = \sup_{n\in\mathbb{N}^*} \mathbf{E}\left((\rho_1^{(n)})^q\right)^n < \infty, \tag{2.16}$$

*for some integer* $q \geq 2$*. Then, for each* $p \in \mathbb{N}^*$ *such that* $1 \leq p < q$*, we have*

$$\lim_{n\to\infty} \mathbf{E}[(\theta_1^{(n)})^p] = \mathbf{E}[(Y_1)^p] = m_p(1),$$

*for the function* $m_p$ *defined in Proposition 2.4.1.*

Before turning to the proof of the theorem, we give an example to illustrate Condition (2.16).

**Example 2.4.3** (Approximation of the moments with NIG log-returns)**.** Take $\ln(\rho_1)$ to be a normal inverse gaussian NIG$(\alpha,\beta,\delta,\mu)$ random variable with $0 \leq |\beta| < \alpha$, $\delta > 0$ and $\mu \in \mathbb{R}$ (recall Example 2.2.6 for the definition). Fix $q \geq 2$. When $n$ is large enough, we can use Taylor's formula, to obtain

$$\sqrt{\alpha^2 - \left(\beta + \frac{q}{\sqrt{n}}\right)^2} = \gamma - \frac{q}{\sqrt{n}}\frac{\beta}{\gamma} - \frac{q^2}{2n}\frac{\alpha^2}{[\alpha^2 - (\beta + x_n)^2]^{3/2}},$$

for some $x_n \in [0, q/\sqrt{n}]$ and thus

$$\lim_{n\to\infty} \mathbf{E}\left(e^{q\gamma_1^{(n)}}\right)^n = \exp\left(q\mu_\rho + \frac{q\delta\alpha^2}{2\gamma^3}\right).$$

Since this limit exists and is finite for each $q \geq 2$, the sequence is bounded and the convergence of the moments depends only on the highest moment of $\xi_1$. Note that the NIG distribution contains the standard Gaussian as a particular case.

*Proof of Theorem 2.4.2.* Since by Corollary 2.2.5, we know that $\theta_1^{(n)} \xrightarrow{d} Y_1$, we have also $(\theta_1^{(n)})^p \xrightarrow{d} (Y_1)^p$, as $n \to \infty$, for $1 \le p < q$. It is thus enough to show that the sequence $((\theta_1^{(n)})^p)_{n\in\mathbb{N}^*}$ is uniformly integrable, which by de la Vallée-Poussin's criterion is implied by the condition $\sup_{n\in\mathbb{N}^*} \mathbf{E}(|\theta_1^{(n)}|^q) < \infty$.

Define $\tilde{R}_t^{(n)} = R_1^{(n)} - R_{1-t}^{(n)}$ and $\tilde{X}_t^{(n)} = X_1^{(n)} - X_{1-t}^{(n)}$ the time-reversed processes of $R^{(n)}$ and $X^{(n)}$ which are defined for $t \in [0,1]$. It is possible to check that $(\tilde{R}_t^{(n)})_{0\le t\le 1} \stackrel{d}{=} (R_t^{(n)})_{0\le t\le 1}$ and $(\tilde{X}_t^{(n)})_{0\le t\le 1} \stackrel{d}{=} (X_t^{(n)})_{0\le t\le 1}$ by checking that the characteristics of these processes are equal (since $[n]-[n(1-t)]-1 = \mathrm{ceil}(nt)-1 = [nt]$, where ceil is the ceiling function), and by applying Theorem II.4.25 p.110 in [Jacod and Shiryaev, 2003]. (See also the example on p.97 in [Jacod and Shiryaev, 2003] for the computation of the characteristics.) Thus, we can imitate the proof of Theorem 3.1. in [Carmona et al., 2001] to obtain

$$\begin{aligned}\theta_1^{(n)} &= e^{R_1^{(n)}} y + \int_{0+}^1 e^{R_1^{(n)} - R_{s-}^{(n)}} dX_s^{(n)} \stackrel{d}{=} e^{\tilde{R}_1^{(n)}} y + \int_{0+}^1 e^{\tilde{R}_{u-}^{(n)}} d\tilde{X}_u^{(n)} \\ &\stackrel{d}{=} e^{R_1^{(n)}} y + \int_{0+}^1 e^{R_{u-}^{(n)}} dX_u^{(n)}.\end{aligned}$$

Then, using the fact that $|a+b|^q \le 2^{q-1}(|a|^q + |b|^q)$, we obtain

$$\mathbf{E}\left(|\theta_1^{(n)}|^q\right) \le 2^{q-1}\left[y^q \mathbf{E}\left(e^{qR_1^{(n)}}\right) + \mathbf{E}\left(\left|\int_{0+}^1 e^{R_{u-}^{(n)}} dX_u^{(n)}\right|^q\right)\right].$$

Denote by $I_1^{(n)}$ and $I_2^{(n)}$ the expectation appearing on the r.h.s. of the above inequality. We will treat each expectation separately.

For $I_1^{(n)}$ we simply have

$$\sup_{n\in\mathbb{N}^*} I_1^{(n)} = \sup_{n\in\mathbb{N}^*} \prod_{i=1}^n \mathbf{E}(e^{q\gamma_i^{(n)}}) = \sup_{n\in\mathbb{N}^*} \mathbf{E}\left(e^{q\gamma_1^{(n)}}\right)^n < \infty. \qquad (2.17)$$

For $I_2^{(n)}$, we start by defining $M_t^{(n)} = \sum_{i=1}^{[nt]} \frac{\ln(\rho_i)-\mu_\rho}{\sqrt{n}}$, for $0 \le t \le 1$. It is possible to check that $(M_t^{(n)})_{0\le t\le 1}$ is a martingale, for each $n \in \mathbb{N}^*$ (for the filtration defined above Theorem 2.2.1.) In fact, the martingale property is checked in the same manner as for $X^{(n)}$ in the proof of Theorem 2.2.1 and the integrability is clear. Thus, $(e^{M_t^{(n)}})_{0\le t\le 1}$ is a submartingale, and using Doob's inequality we obtain

$$\begin{aligned}\mathbf{E}\left(\left|\sup_{0\le t\le 1} e^{R_t^{(n)}}\right|^q\right) &= \left(\sup_{0\le t\le 1} e^{q\mu_\rho \frac{[nt]}{n}}\right)\mathbf{E}\left(\left|\sup_{0\le t\le 1} e^{M_t^{(n)}}\right|^q\right)\\ &\le \max(1, e^{\mu_\rho q})\left(\frac{q}{q-1}\right)^q \mathbf{E}(e^{qM_1^{(n)}})\\ &= \max(1, e^{-\mu_\rho q})\left(\frac{q}{q-1}\right)^q \mathbf{E}(e^{qR_1^{(n)}}).\end{aligned}$$

And so the assumption (2.16) together with (2.17) imply that

$$\sup_{n\in\mathbb{N}^*}\mathbf{E}\left(\left|\sup_{0\le t\le 1} e^{R_t^{(n)}}\right|^q\right) < \infty. \tag{2.18}$$

Writing $X_t^{(n)} = A_t^{(n)} + N_t^{(n)}$, with the processes defined in the proof of Theorem 2.2.1 and using the fact that $|A_t^{(n)}| \le |\mu_\xi| t$, for all $t \ge 0$, we have

$$\begin{aligned}I_2^{(n)} &\le 2^{q-1}\left[\mathbf{E}\left(\left|\int_{0+}^1 e^{R_{s-}^{(n)}} dA_s^{(n)}\right|^q\right) + \mathbf{E}\left(\left|\int_{0+}^1 e^{R_{s-}^{(n)}} dN_s^{(n)}\right|^q\right)\right]\\ &\le 2^{q-1}|\mu_\xi|^q \mathbf{E}\left(\left|\sup_{0\le t\le 1} e^{R_1^{(n)}}\right|^q\right) + 2^{q-1}\mathbf{E}\left(\left|\int_{0+}^1 e^{R_{s-}^{(n)}} dN_s^{(n)}\right|^q\right).\end{aligned}$$

But, from the Burkholder-Davis-Gundy inequality applied twice and the independence of the sequences, we obtain

$$\begin{aligned}\mathbf{E}\left(\left|\int_{0+}^1 e^{R_{s-}^{(n)}} dN_s^{(n)}\right|^q\right) &\le D_q \mathbf{E}\left(\left|\int_{0+}^1 e^{2R_{s-}^{(n)}} d[N^{(n)}]_s\right|^{q/2}\right)\\ &\le D_q \mathbf{E}\left(\sup_{0\le t\le 1} e^{qR_1^{(n)}}\right)\mathbf{E}([N^{(n)}]_1^{q/2})\\ &\le d_q D_q \mathbf{E}\left(\left|\sup_{0\le t\le 1} e^{R_1^{(n)}}\right|^q\right)\mathbf{E}(|N_1^{(n)}|^q)\end{aligned}$$

for some positive constants $D_q$ and $d_q$. Thus,

$$\begin{aligned}\sup_{n\in\mathbb{N}^*} I_2^{(n)} \le 2^{q-1}|\mu_\xi| \sup_{n\in\mathbb{N}^*}\mathbf{E}\left(\left|\sup_{0\le t\le 1} e^{R_t^{(n)}}\right|^q\right)\\ + 2^{q-1} d_q D_q \sup_{n\in\mathbb{N}^*}\mathbf{E}\left(\left|\sup_{0\le t\le 1} e^{R_t^{(n)}}\right|^q\right)\sup_{n\in\mathbb{N}^*}\mathbf{E}\left(|N_1^{(n)}|^q\right),\end{aligned}$$

and so (2.18) implies that it is enough to check that $\sup_{n\in\mathbb{N}^*} \mathbf{E}(|N_1^{(n)}|^q)$ is finite. (Note that a similar argument to the one used to prove the finiteness of $I_2^{(n)}$ is used in a different context in the proof of Lemma 5.1 in [Bankovsky et al., 2011].)

Using the multinomial theorem, we obtain

$$\mathbf{E}(|N_1^{(n)}|^q) = \sum_{k_1+\cdots+k_n=q} \binom{q}{k_1,\ldots,k_n} \prod_{i=1}^{n} \mathbf{E}\left(\left(\frac{\xi_1-\mu_\xi}{\sqrt{n}}\right)^{k_i}\right),$$

where the sum is taken over all non-negative integer solutions of $k_1+\cdots+k_n = q$. Since $\mathbf{E}((\xi_1-\mu_\xi)^{k_i}) = 0$, when $k_i = 1$, we can sum over the partitions with $k_i \neq 1$ for all $i = 1,\ldots,n$. Now, since $k_i/q \leq 1$, we have by Jensen's inequality

$$\begin{aligned}\prod_{i=1}^{n} \mathbf{E}\left(\left(\frac{\xi_1-\mu_\xi}{\sqrt{n}}\right)^{k_i}\right) &\leq \prod_{i=1}^{n} \mathbf{E}\left(\left|\frac{\xi_1-\mu_\xi}{\sqrt{n}}\right|^q\right)^{\frac{k_i}{q}} \\ &= n^{-q/2}\mathbf{E}(|\xi_1-\mu_\xi|^q).\end{aligned}$$

Thus,

$$I_3^{(n)} \leq 2^{q-1}\mathbf{E}(|\xi_1-\mu_\xi|^q)n^{-q/2} \sum_{k_1+\cdots+k_n=q,k_i\neq 1} \binom{q}{k_1,\ldots,k_n}.$$

and since we need to take the supremum over $n \in \mathbb{N}^*$, we need to check that the r.h.s. of the above inequality is bounded in $n$. For this, note that the sum of multinomial coefficients is equal to

$$\sum_{i=1}^{[q/2]} \binom{n}{i} \sum_{l_1+\cdots+l_i=q-2i} \binom{q}{l_1+2,\ldots,l_i+2}, \tag{2.19}$$

where, for each $i = 1,\ldots,[q/2]$, the second sum is taken over all non-negative integer solutions of $l_1+\cdots+l_i = q-2i$. This follows from the fact that if $(k_1,\ldots,k_n)$ are non-negative integer solutions of $k_1+\cdots+k_n = q$, we have, since $k_i \neq 1$, that the number of non-zero terms in $(k_1,\ldots,k_n)$ is at most $[q/2]$. Thus, letting $i$ be the number of non-zero terms and $(j_1,\ldots,j_i)$ their indices, we find that $(k_{j_1}-2)+\cdots+(k_{j_i}-2) = q-2i$, which yields the claimed equality. Then, we have

$$\binom{q}{l_1+2,\ldots,l_i+2} \leq C_i \binom{q-2i}{l_1,\ldots,l_i},$$

with $C_i = 2^{-i}q!/(q-2i)!$ and

$$\sum_{l_1+\cdots+l_i=q-2i} \binom{q-2i}{l_1,\ldots,l_i} = i^{q-2i}.$$

Let $C_q = \max_{i=1,\ldots,[q/2]} C_i$ and $K_q = 2^{q-1}C_q\mathbf{E}(|\xi_1 - \mu_\xi|^q)$, remark that the binomial coefficient in (2.19) is bounded by $n^{[q/2]}$ and that

$$I_3^{(n)} \leq K_q n^{-q/2} \sum_{i=1}^{[q/2]} \binom{n}{i} i^{q-2i} \leq K_q n^{-q/2+[q/2]} \sum_{i=1}^{[q/2]} i^{q-2i},$$

which is bounded in $n$. Thus, $\sup_{n\in\mathbb{N}^*} I_3^{(n)} < \infty$, $\sup_{n\in\mathbb{N}^*} I_2^{(n)} < \infty$ and

$$\sup_{n\in\mathbb{N}^*} \mathbf{E}\left(|\theta_1^{(n)}|^q\right) \leq 2^{q-1}y^q \sup_{n\in\mathbb{N}^*} I_1^{(n)} + 2^{q-1} \sup_{n\in\mathbb{N}^*} I_2^{(n)} < \infty.$$

So, the sequence $((\theta_1^{(n)})^p)_{n\in\mathbb{N}^*}$ is uniformly integrable and we have $\lim_{n\to\infty} \mathbf{E}[(\theta_1^{(n)})^p] = \mathbf{E}[(Y_1)^p]$. □

# Chapter 3

# On the Ruin Problem for GOU Processes

In this chapter[1], we study the ruin problem for the GOU process in a general framework where the investment process $R$ is a semimartingale. After introducing our assumptions, we recall some related results in Section 3.1. In Section 3.2 we obtain upper bounds on the finite and infinite time ruin probabilities and, in Section 3.3, lower bounds for these quantities as well as the logarithmic asymptotic for them. The results of these sections depends on the computation of some quantity for which we give an explicit method when $R$ is a Lévy process, in Section 3.4. Finally, in Section 3.5, we give sufficient conditions on $R$ for ruin with probability one and, in the Lévy case, show that they correspond to the known condition based on the characteristic triplet of $R$.

Classically, the value of an insurance company with initial capital $y > 0$ and investing in a financial market, denoted by $Y = (Y_t)_{t\geq 0}$, is given as the solution of the following linear stochastic differential equation

$$Y_t = y + X_t + \int_{0+}^{t} Y_{s-} dR_s, \;\; t \geq 0, \tag{3.1}$$

where $X = (X_t)_{t\geq 0}$ and $R = (R_t)_{t\geq 0}$ are two independent stochastic processes

[1]This chapter is based on the joint work done with Lioudmila Vostrikova for the paper [Spielmann and Vostrikova, 2019] which is accepted for publication in *Theory of Probability and its Applications.*

defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and chosen so that (3.1) makes sense. The process $X$ then represents the profit and loss of the business activity and $R$ represents the return of the investment. The ruin problem in this general setting was first studied in [Paulsen, 1993].

In this chapter, we assume again that the processes $X = (X_t)_{t\geq 0}$ and $R = (R_t)_{t\geq 0}$ are independent and such that $X$ is a Lévy process and $R$ is a semi-martingale. We suppose also that the jumps of $R$ denoted $\Delta R_t = R_t - R_{t-}$ are strictly bigger than $-1$, for all $t > 0$. In this case, from Proposition 1.3.3, we have $[X, R]_t = 0$, for all $t \geq 0$, and that the equation (3.1) admits a unique strong solution given by

$$Y_t = \mathcal{E}(R)_t \left( y + \int_{0+}^t \frac{dX_s}{\mathcal{E}(R)_{s-}} \right), \ \ t \geq 0 \tag{3.2}$$

where $\mathcal{E}(R)$ is Doléans-Dade's exponential. Moreover, since $\Delta R_t > -1$, for all $t \geq 0$ implies that $\mathcal{E}(R)_t > 0$, for all $t \geq 0$, we obtain

$$\begin{aligned} &\mathbf{P}\left( \sup_{0\leq t\leq T} \left( - \int_{0+}^t \frac{dX_s}{\mathcal{E}(R)_{s-}} \right) > y \right) \leq \mathbf{P}(\tau(y) \leq T) \\ &\quad \leq \mathbf{P}\left( \sup_{0\leq t\leq T} \left( - \int_{0+}^t \frac{dX_s}{\mathcal{E}(R)_{s-}} \right) \geq y \right). \end{aligned} \tag{3.3}$$

Thus, the ruin problem for the GOU is strongly related to the first passage time of $-\int_{0+}^{\cdot} \frac{dX_s}{\mathcal{E}(R)_{s-}}$.

In fact, we will show in this chapter that the behaviour of the first passage time of this stochastic integral depends strongly on the behaviour of the exponential functional of $R$ at $T$, i.e. on the behaviour of

$$I_T = \int_0^T e^{-\hat{R}_s} ds \ \text{ and } \ J_T(\alpha) = \int_0^T e^{-\alpha \hat{R}_s} ds$$

where $\alpha > 0$ and $\hat{R}_t = \ln \mathcal{E}(R)_t$, for all $t \geq 0$, is the exponential transform of $R$ or on the *exponential functional* of $R$,

$$I_\infty = \int_0^\infty e^{-\hat{R}_s} ds \ \text{ and } \ J_\infty(\alpha) = \int_0^\infty e^{-\alpha \hat{R}_s} ds.$$

For convenience, we let $J_T = J_T(2)$ and $J_\infty = J_\infty(2)$.

In the following, we denote the generating triplet of the Lévy process $X$ by $(a_X, \sigma_X^2, K_X)$ where $a_X \in \mathbb{R}$, $\sigma_X \geq 0$ and $K_X$ is a Lévy measure. We recall that, using the Lévy-Itô decomposition, the process $X$ can then be written in the form

$$\begin{aligned} X_t = a_X t + \sigma_X W_t &+ \int_{0+}^{t} \int_{\{|x| \leq 1\}} x(\mu_X(ds, dx) - K_X(dx)ds) \\ &+ \int_{0+}^{t} \int_{\{|x|>1\}} x \mu_X(ds, dx), \end{aligned} \tag{3.4}$$

where $\mu_X$ is the jump measure of $X$, $W$ is standard Brownian Motion and the processes appearing in this decomposition are independent. We recall also that $R = (R_t)_{t \geq 0}$ can similarly be defined by its semimartingale decomposition

$$\begin{aligned} R_t = B_t + R_t^c &+ \int_{0+}^{t} \int_{\{|x| \leq 1\}} x(\mu_R(ds, dx) - \nu_R(ds, dx)) \\ &+ \int_{0+}^{t} \int_{\{|x|>1\}} x \mu_R(ds, dx), \end{aligned} \tag{3.5}$$

where $B = (B_t)_{t \geq 0}$ is a drift part, $R^c = (R_t^c)_{t \geq 0}$ is the continuous martingale part of $R$, $\mu_R$ is the jump measure of $R$ and $\nu_R$ is its compensator.

## 3.1 Related Results

The ruin problem for the GOU process is well studied, since, as we mentioned in the introduction, these processes are used to model the surplus process of an insurance company facing both insurance and market risks. Thus, before describing our our results, we give a brief review of the related literature.

The special case when $R_t = rt$, with $r > 0$, for all $t \geq 0$ (non-risky investment) is well known and we refer to [Paulsen, 2008] and references therein for the main results. In brief, in that case and under some additional conditions, the ruin probability decreases even faster than an exponential since the capital of the insurance company is constantly increasing.

The case of risky investment is also well-studied. In that case, it is assumed in general that $X$ and $R$ are independent Lévy processes. The first results in this setting appear in [Kalashnikov and Norberg, 2002] (and later in [Yuen et al., 2004]) where it was shown that under some conditions there exists $C > 0$ and $y_0 \geq 0$ such that for all $y \geq y_0$ and for some $b > 0$

$$\mathbf{P}(\tau(y) < \infty) \geq Cy^{-b}.$$

Qualitatively, this means that the ruin probability cannot decrease faster as a power function, i.e. the de-growth is much slower than in the no-investment case. Later, under some conditions on the Lévy triplets of $X$ and $R$, it was shown in [Paulsen, 2002] that for some $\beta > 0$ and $\epsilon > 0$, there exists $C > 0$ such that, as $y \to \infty$,

$$y^{\beta}\,\mathbf{P}(\tau(y) < \infty) = C + o(y^{-\epsilon}).$$

In [Kabanov and Pergamentshchikov, 2016], it was proven, under different assumptions on the Lévy triplets and when $X$ has no negative jumps, that there exists $C > 0$ such that for the above $\beta > 0$

$$\lim_{y\to\infty} y^{\beta}\,\mathbf{P}(\tau(y) < \infty) = C.$$

Results concerning bounds on $\mathbf{P}(\tau(y) < +\infty)$ are given in [Kalashnikov and Norberg, 2002] where it is shown that, for all $\epsilon > 0$, there exists $C > 0$ such that for all $y \geq 0$ and the same $\beta > 0$

$$\mathbf{P}(\tau(y) < \infty) \leq Cy^{-\beta+\epsilon}.$$

In less general settings similar results are available. The case when $X$ is a compound Poisson process with drift and exponential jumps and $R$ is a Brownian motion with drift is studied in [Frolova et al., 2002] (negative jumps only) and in [Kabanov and Pergamentshchikov, 2018] (positive jumps only). In [Pergamentshchikov and Zeitouny, 2006] the model with negative jumps is generalized to the case where the drift of $X$ is a bounded stochastic process.

Some exact results for the ultimate ruin probability are available in specific models (see e.g. [Paulsen, 2008] and [Yuen et al., 2004]) and conditions for ruin with probability one are given, for different levels of generality, in

[Frolova et al., 2002], [Kabanov and Pergamentshchikov, 2016], [Kabanov and Pergamentshchikov, 2018], [Kalashnikov and Norberg, 2002], [Paulsen, 1998] and [Pergamentshchikov and Zeitouny, 2006].

We see also that in the mentioned reference the focus was on the case when $R$ is a Lévy process. We believe that, even if the results of the last chapter suggest a model where $R$ is a Lévy process, the return on investment cannot be modelled in general by a homogeneous process. Indeed, the market conditions can change over time or switch between different states. Up to our knowledge, the only paper that focuses also on a semimartingale market is [Hult and Lindskog, 2011] where the asymptotic as $y \to \infty$ of the finite time ruin probability are studied when $X$ has regularly varying tails and $R$ is a semimartingale satisfying some conditions.

We see that these results are concerned mainly with the ultimate ruin probability or its asymptotic when $y \to \infty$. We believe that both these restrictions invite criticism. The first one by noting that, from a practical point of view, one is interested in the possibility of ruin before a certain time, e.g. before reaching retirement, and not in the possibility of ultimate ruin. The focus on the asymptotic when $y \to \infty$ means that it is assumed that the insurance company has a very large capital, which is not the usual situation when focusing on the possibility of ruin. That is why in addition to letting $R$ be a semimartingale we focus, in this chapter, on inequalities for $\mathbf{P}(\tau(y) \leq T)$ which hold for every $y > 0$ and we will see that this allows us to recover some known results on the ultimate probability of ruin and on its asymptotic.

## 3.2 Upper Bound for the Ruin Probabilities

Define

$$\beta_T = \sup\left\{\beta \geq 0 : \mathbf{E}(J_T^{\beta/2}) < \infty, \mathbf{E}(J_T(\beta)) < \infty\right\}.$$

The main result of this section is the following.

**Theorem 3.2.1.** *Let $T > 0$. Assume that $\beta_T > 0$ and that, for some $0 < \alpha < \beta_T$, we have*

$$\int_{\{|x|>1\}} |x|^\alpha K_X(dx) < \infty \ \text{(or equivalently } \mathbf{E}(|X_1|^\alpha) < \infty). \tag{3.6}$$

*Then, for all $y > 0$,*

$$\mathbf{P}(\tau(y) \leq T) \leq \frac{C_1\mathbf{E}(I_T^\alpha) + C_2\mathbf{E}(J_T^{\alpha/2}) + C_3\mathbf{E}(J_T(\alpha))}{y^\alpha}, \tag{3.7}$$

*where the expectations on the right hand side are finite and $C_1 \geq 0$, $C_2 \geq 0$, and $C_3 \geq 0$ are constants that depend only on $\alpha$ in an explicit way. Moreover, if* (3.6) *holds for all $0 < \alpha < \beta_T$, then*

$$\limsup_{y\to\infty} \frac{\ln\left(\mathbf{P}(\tau(y) \leq T)\right)}{\ln(y)} \leq -\beta_T. \tag{3.8}$$

This theorem is, up to our knowledge, the first result establishing an upper bound, when $R$ is not deterministic, for the ruin probability before a finite time even in the case when $R$ is a Lévy process. The theorem indicates that, when $\beta_T = \infty$, under the mentioned conditions, the ruin probability decreases faster than any power function and that, when $\beta_T < \infty$, the ruin probability decreases at least as a power function as $y \to \infty$.

As a corollary, we can obtain a similar result for the ultimate ruin probability. Let

$$\beta_\infty = \sup\left\{\beta \geq 0 : \mathbf{E}(I_\infty^\beta) < \infty, \mathbf{E}(J_\infty^{\beta/2}) < \infty, \mathbf{E}(J_\infty(\beta)) < \infty\right\}.$$

Since $(I_t)_{t\geq 0}$, $(J_t)_{t\geq 0}$ and $(J_t(\alpha))_{t\geq 0}$ are increasing we obtain the following by letting $T \to \infty$ and using the monotone convergence theorem with the upper bound of Theorem 3.2.1.

**Corollary 3.2.2.** *Assume that $\beta_\infty > 0$ and that the condition (3.6) holds for some $0 < \alpha < \beta_\infty$, then*

$$\mathbf{P}(\tau(y) < \infty) \leq \frac{C_1\mathbf{E}(I_\infty^\alpha) + C_2\mathbf{E}(J_\infty^{\alpha/2}) + C_3\mathbf{E}(J_\infty(\alpha))}{y^\alpha},$$

*where $C_1 \geq 0$, $C_2 \geq 0$, and $C_3 \geq 0$ are constants that depend only on $\alpha$ in an explicit way. Moreover, if* (3.6) *holds for all $0 < \alpha < \beta_\infty$, then*

$$\limsup_{y\to\infty} \frac{\ln\left(\mathbf{P}(\tau(y) < \infty)\right)}{\ln(y)} \leq -\beta_\infty.$$

Before giving the proof of Theorem 3.2.1, we now state some preliminary results. The following lemma proves that the expectation on the r.h.s. of (3.7) are indeed finite.

**Lemma 3.2.3.** *For all $T > 0$, we have the following.*

*(a) If $0 < \alpha < 2$, $\mathbf{E}(J_T^{\alpha/2}) < \infty$ implies $\mathbf{E}(I_T^{\alpha}) < \infty$ and $\mathbf{E}(J_T(\alpha)) < \infty$.*

*(b) If $\alpha \geq 2$, $\mathbf{E}(J_T(\alpha)) < \infty$ implies $\mathbf{E}(I_T^{\alpha}) < \infty$ and $\mathbf{E}(J_T^{\alpha/2}) < \infty$.*

*Proof.* First note that by the Cauchy-Schwarz inequality we obtain, for all $T > 0$,

$$I_T = \int_0^T \mathcal{E}(R)_s^{-1} ds \leq \sqrt{T} \left( \int_0^T \mathcal{E}(R)_s^{-2} ds \right)^{1/2} = \sqrt{T}\sqrt{J_T}.$$

So, $\mathbf{E}(I_T^{\alpha}) \leq T^{\alpha/2} \mathbf{E}(J_T^{\alpha/2})$, for all $\alpha > 0$.

Now, if $0 < \alpha < 2$, we have $\frac{2}{\alpha} > 1$ and by Hölder's inequality

$$J_T(\alpha) = \int_0^T \mathcal{E}(R)_s^{-\alpha} ds \leq T^{(2-\alpha)/2} \left( \int_0^T \mathcal{E}(R)_s^{-2} ds \right)^{\alpha/2} = T^{(2-\alpha)/2} J_T^{\alpha/2}.$$

These inequalities yield (a).

Now, if $\alpha \geq 2$, we have either $\alpha = 2$ which yields the desired result or $\alpha > 2$. In that case, we have $\frac{\alpha}{2} > 1$ and, by Hölder's inequality, we obtain

$$\begin{aligned} J_T &= \int_0^T \mathcal{E}(R)_s^{-2} ds \leq T^{(\alpha-2)/\alpha} \left( \int_0^T \mathcal{E}(R)_s^{-\alpha} ds \right)^{2/\alpha} \\ &= T^{(\alpha-2)/\alpha} J_T(\alpha)^{2/\alpha}. \end{aligned}$$

So, $\mathbf{E}(J_T^{\alpha/2}) \leq T^{(\alpha-2)/2} \mathbf{E}(J_T(\alpha))$, which yields (b). □

The main ingredient for the proof of Theorem 3.2.1 will the following decomposition of the stochastic integral process into different parts using the Lévy-Itô decomposition. Denote by $M^d = (M_t^d)_{t\geq 0}$ the local martingale defined as:

$$M_t^d = \int_{0+}^t \int_{\{|x|\leq 1\}} \frac{x}{\mathcal{E}(R)_{s-}} (\mu_X(ds, dx) - K_X(dx)ds)$$

and by $U = (U_t)_{t\geq 0}$ the process given by

$$U_t = \int_{0+}^t \int_{\{|x|>1\}} \frac{x}{\mathcal{E}(R)_{s-}} \mu_X(ds, dx).$$

If $\int_{\{|x|>1\}} |x| K_X(dx) < \infty$, we can also define the local martingale $N^d = (N_t^d)_{t\geq 0}$ as

$$N_t^d = \int_{0+}^t \int_{\mathbb{R}} \frac{x}{\mathcal{E}(R)_{s-}} (\mu_X(ds, dx) - K_X(dx)ds).$$

**Proposition 3.2.4.** *We have the following identity in law:*

$$\left(\int_{0+}^t \frac{dX_s}{\mathcal{E}(R)_{s-}}\right)_{t\geq 0} \stackrel{d}{=} \left(a_X I_t + \sigma_X W_{J_t} + M_t^d + U_t\right)_{t\geq 0}.$$

*Moreover, if* $\int_{\{|x|>1\}} |x| K_X(dx) < \infty$ *(or equivalently* $\mathbf{E}(|X_1|) < \infty$*), then,*

$$\left(\int_{0+}^t \frac{dX_s}{\mathcal{E}(R)_{s-}}\right)_{t\geq 0} \stackrel{d}{=} \left(\delta_X I_t + \sigma_X W_{J_t} + N_t^d\right)_{t\geq 0},$$

*where* $\delta_X = \mathbf{E}(X_1) = a_X + \int_{\{|x|>1\}} x K_X(dx)$.

*Proof.* We show first that

$$\mathcal{L}\left(\left(\int_{0+}^t \frac{dX_s}{\mathcal{E}(R)_{s-}}\right)_{t\geq 0} \mid \mathcal{E}(R)_s = q_s, s \geq 0\right) = \mathcal{L}\left(\left(\int_{0+}^t \frac{dX_s}{q_{s-}}\right)_{t\geq 0}\right)$$

To prove this equality in law we consider the representation of the stochastic integrals by Riemann sums (see [Jacod and Shiryaev, 2003], Proposition I.4.44, p. 51). We recall that for any increasing sequence of stopping times $\tau = (T_n)_{n\in\mathbb{N}}$ with $T_0 = 0$ such that $\sup_n T_n = \infty$ and $T_n < T_{n+1}$ on the set $\{T_n < \infty\}$, the Riemann approximation of the stochastic integral $\int_0^t \frac{dX_s}{\mathcal{E}(R)_{s-}}$ will be

$$\tau\left(\int_{0+}^t \frac{dX_s}{\mathcal{E}(R)_{s-}}\right) = \sum_{n=0}^{\infty} \frac{1}{\mathcal{E}(R)_{T_n-}} \left(X_{T_{n+1}\wedge t} - X_{T_n \wedge t}\right)$$

The sequence $\tau_n = (T(n,m))_{m\in\mathbb{N}}$ of adapted subdivisions is called Riemann sequence if $\sup_{m\in\mathbb{N}}(T(n,m+1)\wedge t - T(n,m)\wedge t) \to 0$ as $n \to \infty$ for all

$t > 0$. For our purposes we will take a deterministic Riemann sequence. Then, Proposition I.4.44, p.51 of [Jacod and Shiryaev, 2003] says that for all $t > 0$

$$\tau_n\left(\int_{0+}^{t} \frac{dX_s}{\mathcal{E}(R)_{s-}}\right) \xrightarrow{\mathbf{P}} \int_{0+}^{t} \frac{dX_s}{\mathcal{E}(R)_{s-}} \tag{3.9}$$

and

$$\tau_n\left(\int_{0+}^{t} \frac{dX_s}{q_{s-}}\right) \xrightarrow{\mathbf{P}} \int_{0+}^{t} \frac{dX_s}{q_{s-}} \tag{3.10}$$

where $\xrightarrow{\mathbf{P}}$ denotes the convergence in probability. According to the Kolmogorov theorem, the law of the process is entirely defined by its finite-dimensional distributions. Let us take for $k \geq 0$ a subdivision $t_0 = 0 < t_1 < t_2 \cdots < t_k$ and a continuous bounded function $F : \mathbb{R}^k \to \mathbb{R}$, to prove by standard arguments that

$$\begin{aligned}
&\mathbf{E}\left[F\left(\tau_n\left(\int_{0+}^{t_1} \frac{dX_s}{\mathcal{E}(R)_{s-}}\right), \cdots \tau_n\left(\int_{0+}^{t_k} \frac{dX_s}{\mathcal{E}(R)_{s-}}\right)\right) \mid \mathcal{E}(R)_s = q_s, s \geq 0\right] \\
&\qquad = \mathbf{E}\left[F\left(\tau_n\left(\int_{0+}^{t_1} \frac{dX_s}{q_{s-}}\right), \cdots \tau_n\left(\int_{0+}^{t_k} \frac{dX_s}{q_{s-}}\right)\right)\right]
\end{aligned}$$

Taking into account (3.9) and (3.10), we pass to the limit as $n \to \infty$ and we obtain

$$\begin{aligned}
&\mathbf{E}\left[F\left(\int_{0+}^{t_1} \frac{dX_s}{\mathcal{E}(R)_{s-}}, \cdots \int_{0+}^{t_k} \frac{dX_s}{\mathcal{E}(R)_{s-}}\right) \mid \mathcal{E}(R)_s = q_s, s \geq 0\right] \\
&\qquad = \mathbf{E}\left[F\left(\int_{0+}^{t_1} \frac{dX_s}{q_{s-}}, \cdots \int_{0+}^{t_k} \frac{dX_s}{q_{s-}}\right)\right]
\end{aligned}$$

and this proves the claim.

Using the decomposition (3.4) we get that

$$\begin{aligned}
\int_{0+}^{t} \frac{dX_s}{q_{s-}} = a_X \int_{0}^{t} \frac{ds}{q_s} + \sigma_X \int_{0+}^{t} \frac{dW_s}{q_{s-}} \\
+ \int_{0+}^{t} \int_{\{|x| \leq 1\}} \frac{x}{q_{s-}} (\mu_X(ds, dx) - K_X(dx)ds) \\
+ \int_{0+}^{t} \int_{\{|x| > 1\}} \frac{x}{q_{s-}}\, \mu_X(ds, dx).
\end{aligned}$$

We denote the last two terms in the r.h.s. of the equality above by $M_t^d(q)$ and $U_t(q)$ respectively. Recall that since $X$ is Lévy process the four processes appearing in the right-hand side of the above equality are independent. We use the well-known identity in law

$$\left(\int_{0+}^t \frac{dW_s}{q_{s-}}\right)_{t\geq 0} \stackrel{d}{=} \left(W_{\int_0^t \frac{ds}{q_s^2}}\right)_{t\geq 0}$$

to write

$$\begin{aligned}\left(a_X \int_0^t \frac{ds}{q_s},\ \sigma_X \int_{0+}^t \frac{dW_s}{q_{s-}}, M_t^d(q), U_t(q)\right)_{t\geq 0}\\ \stackrel{d}{=} \left(a_X \int_0^t \frac{ds}{q_s}, \sigma_X W_{\int_0^t \frac{ds}{q_s^2}}, M_t^d(q), U_t(q)\right)_{t\geq 0}.\end{aligned}$$

Then, we take the sum of these processes and we integrate w.r.t. the law of $\mathcal{E}(R)$. This yields the first result.

The proof of the second part is the same except we take the following decomposition of $X$:

$$X_t = \delta_X t + \sigma_X W_t + \int_{0+}^t \int_{\mathbb{R}} x(\mu_X(ds, dx) - K_X(dx)ds).$$

$\square$

Thus, the study of the stochastic integral can be reduced to the study of the elements in the decomposition given in Proposition 3.2.4. To treat the compensated integral term in this decomposition, we will use Novikov-Bichteler-Jacod maximal inequalities (see [Novikov, 1975], [Bichteler and Jacod, 1983] and also [Marinelli and Röckner, 2014]) which we will state below after introducing some notations.

Let $f : (\omega, t, x) \mapsto f(\omega, t, x)$ be a left-continuous and measurable random function on $\Omega \times \mathbb{R}_+ \times \mathbb{R}$. Specializing the notations of [Novikov, 1975] to our case, we say that $f \in F_2$ if, for almost all $\omega \in \Omega$,

$$\int_0^t \int_{\mathbb{R}} f^2(\omega, s, x) K_X(dx) ds < \infty.$$

If $f \in F_2$, we can define the compensated integral by

$$C_f(t) = \int_{0+}^{t} \int_{\mathbb{R}} f(\omega, s, x) \left(\mu_X(ds, dx) - K_X(dx)ds\right)$$

for all $t \geq 0$, as an usual stochastic integral and it is possible to show that the definition coincides with the one given in Section 1.2.3 when $f \in F_2$ satisfies additionally the conditions in that section.

**Proposition 3.2.5** (Novikov-Bichteler-Jacod inequalities, c.f. Theorem 1 in [Novikov, 1975])**.** *Let $f$ be a left-continuous measurable random function with $f \in F_2$. Let $C_f = (C_f(t))_{t\geq 0}$ be the compensated integral of $f$ as defined above.*

*(a) For all $0 \leq \alpha \leq 2$,*

$$\mathbf{E}\left(\sup_{0\leq t\leq T} |C_f(t)|^\alpha\right) \leq K_1 \mathbf{E}\left[\left(\int_0^T \int_{\mathbb{R}} f^2 K_X(dx)ds\right)^{\alpha/2}\right].$$

*(b) For all $\alpha \geq 2$,*

$$\begin{aligned}\mathbf{E}\left(\sup_{0\leq t\leq T} |C_f(t)|^\alpha\right) &\leq K_2 \mathbf{E}\left[\left(\int_0^T \int_{\mathbb{R}} |f|^2 K_X(dx)ds\right)^{\alpha/2}\right] \\ &\quad + K_3 \mathbf{E}\left(\int_0^T \int_{\mathbb{R}} |f|^\alpha K_X(dx)ds\right)\end{aligned}$$

*where $K_1 \geq 0$, $K_2 \geq 0$, and $K_3 \geq 0$ are constants depending only on $\alpha$ in an explicit way.*

We are now ready for the proof of Theorem 3.2.1.

*Proof of Theorem 3.2.1.* Note that

$$\begin{aligned}&\sup_{0\leq t\leq T} -(a_X I_t + \sigma_X W_{J_t} + M_t^d + U_t) \\ &\qquad \leq |a_X| I_T + \sup_{0\leq t\leq T} \sigma_X |W_{J_t}| + \sup_{0\leq t\leq T} |M_t^d| + \sup_{0\leq t\leq T} |U_t|,\end{aligned}$$

and that for positive random variables $Z_1, Z_2, Z_3, Z_4$ we have

$$\begin{aligned}&\{Z_1 + Z_2 + Z_3 + Z_4 \geq y\}\\ &\qquad \subseteq \left\{Z_1 \geq \frac{y}{4}\right\} \cup \left\{Z_2 \geq \frac{y}{4}\right\} \cup \left\{Z_3 \geq \frac{y}{4}\right\} \cup \left\{Z_4 \geq \frac{y}{4}\right\}.\end{aligned}$$

Therefore, using Equation (3.3) and Proposition 3.2.4, we obtain

$$\begin{aligned}\mathbf{P}(\tau(y) \leq T) \leq& \mathbf{P}\left(\sup_{0\leq t\leq T} -(a_X I_t + \sigma_X W_{J_t} + M_t^d + U_t) \geq y\right)\\ \leq& \mathbf{P}\left(|a_X| I_T \geq \frac{y}{4}\right) + \mathbf{P}\left(\sup_{0\leq t\leq T} \sigma_X |W_{J_t}| \geq \frac{y}{4}\right)\\ &+ \mathbf{P}\left(\sup_{0\leq t\leq T} |M_t^d| \geq \frac{y}{4}\right) + \mathbf{P}\left(\sup_{0\leq t\leq T} |U_t| \geq \frac{y}{4}\right).\end{aligned}$$

For the first term, using Markov's inequality, we obtain

$$\mathbf{P}\left(|a_X| I_T \geq \frac{y}{4}\right) \leq \frac{4^\alpha |a_X|^\alpha}{y^\alpha} \mathbf{E}(I_T^\alpha).$$

For the second term, since $(J_t)_{0\leq t\leq T}$ is increasing we can change the time in the supremum and condition on $(\mathcal{E}(R)_t)_{0\leq t\leq T}$ to obtain

$$\begin{aligned}\mathbf{P}\left(\sup_{0\leq t\leq T} \sigma_X |W_{J_t}| \geq \frac{y}{4}\right) &= \mathbf{P}\left(\sup_{0\leq t\leq J_T} \sigma_X |W_t| \geq \frac{y}{4}\right)\\ &= \mathbf{E}\left[\mathbf{P}\left(\sup_{0\leq t\leq J_T} \sigma_X |W_t| \geq \frac{y}{4} \middle| (\mathcal{E}(R)_t)_{0\leq t\leq T}\right)\right]\end{aligned}$$

Since $W$ and $R$ are independent, we obtain, using the reflection principle, the fact that $W_{\int_0^T q_t^{-2} dt} \stackrel{d}{=} \left(\int_0^T q_t^{-2} dt\right)^{1/2} W_1$ and Markov's inequality, that

$$\begin{aligned}&\mathbf{P}\left(\sup_{0\leq t\leq J_T} \sigma_X |W_t| \geq \frac{y}{4} \middle| \mathcal{E}(R)_t = q_t, 0 \leq t \leq T\right)\\ &= 2\mathbf{P}\left(\left(\int_0^T q_t^{-2} dt\right)^{1/2} \sigma_X |W_1| \geq \frac{y}{4}\right)\\ &\leq 2\frac{4^\alpha \sigma_X^\alpha}{y^\alpha} \left(\int_0^T q_t^{-2} dt\right)^{\alpha/2} \mathbf{E}(|W_1|^\alpha).\end{aligned}$$

Then, since $\mathbf{E}(|W_1|^\alpha) = \frac{2^{\alpha/2}}{\sqrt{\pi}}\Gamma\left(\frac{\alpha+1}{2}\right)$, we obtain

$$\mathbf{P}\left(\sup_{0\le t\le T}\sigma_X|W_{J_t}| \ge \frac{y}{4}\right) \le \frac{2^{(5\alpha+2)/2}\Gamma\left(\frac{\alpha+1}{2}\right)\sigma_X^\alpha}{\sqrt{\pi}y^\alpha}\mathbf{E}(J_T^{\alpha/2}).$$

Note that the inequalities for the first two terms work for all $\alpha > 0$.

Suppose now that $0 < \alpha \le 1$. We see that $\mathcal{E}(R)_{t-}^{-1}x\mathbf{1}_{\{|x|\le 1\}} \in F_2$. Therefore, using Markov's inequality and part (a) of Proposition 3.2.5, we obtain

$$\begin{aligned}
\mathbf{P}\left(\sup_{0\le t\le T}|M_t^d| \ge \frac{y}{4}\right) &\le \frac{4^\alpha}{y^\alpha}\mathbf{E}\left(\sup_{0\le t\le T}|M_t^d|^\alpha\right)\\
&\le K_1\frac{4^\alpha}{y^\alpha}\mathbf{E}\left[\left(\int_0^T\int_{\mathbb{R}}\frac{x^2}{\mathcal{E}(R)_{s-}^2}\mathbf{1}_{\{|x|\le 1\}}K_X(dx)ds\right)^{\alpha/2}\right]\\
&= K_1\frac{4^\alpha}{y^\alpha}\left(\int_{\mathbb{R}}x^2\mathbf{1}_{\{|x|\le 1\}}K_X(dx)\right)^{\alpha/2}\mathbf{E}(J_T^{\alpha/2}).
\end{aligned}$$

For the last term, note that since $0 < \alpha \le 1$, we have $\left(\sum_{i=1}^N x_i\right)^\alpha \le \sum_{i=1}^N x_i^\alpha$, for $x_i \ge 0$ and $N \in \mathbb{N}^*$ and, for each $t \ge 0$,

$$\begin{aligned}
|U_t|^\alpha &\le \left(\sum_{0<s\le t}\mathcal{E}(R)_{s-}^{-1}|\Delta X_s|\mathbf{1}_{\{|\Delta X_s|>1\}}\right)^\alpha\\
&\le \sum_{0<s\le t}\mathcal{E}(R)_{s-}^{-\alpha}|\Delta X_s|^\alpha\mathbf{1}_{\{|\Delta X_s|>1\}}\\
&= \int_{0+}^t\int_{\mathbb{R}}\mathcal{E}(R)_{s-}^{-\alpha}|x|^\alpha\mathbf{1}_{\{|x|>1\}}\mu_X(ds,dx).
\end{aligned}$$

Therefore, using Markov's inequality and the compensation formula, we obtain

$$\begin{aligned}
\mathbf{P}\left(\sup_{0\le t\le T}|U_t| \ge \frac{y}{4}\right) &\le \frac{4^\alpha}{y^\alpha}\mathbf{E}\left(\sup_{0\le t\le T}|U_t|^\alpha\right)\\
&\le \frac{4^\alpha}{y^\alpha}\mathbf{E}\left(\sup_{0\le t\le T}\int_{0+}^t\int_{\mathbb{R}}\mathcal{E}(R)_{s-}^{-\alpha}|x|^\alpha\mathbf{1}_{\{|x|>1\}}\mu_X(ds,dx)\right)\\
&= \frac{4^\alpha}{y^\alpha}\mathbf{E}\left(\int_0^T\int_{\mathbb{R}}\mathcal{E}(R)_{s-}^{-\alpha}|x|^\alpha\mathbf{1}_{\{|x|>1\}}K_X(dx)ds\right)\\
&= \frac{4^\alpha}{y^\alpha}\left(\int_{\mathbb{R}}|x|^\alpha\mathbf{1}_{\{|x|>1\}}K_X(dx)\right)\mathbf{E}(J_T(\alpha)).
\end{aligned}$$

This finishes the proof when $0 < \alpha \leq 1$.

Suppose now that $1 < \alpha \leq 2$. The bound for $\mathbf{P}\left(\sup_{0\leq t\leq T}|M_t^d| \geq \frac{y}{4}\right)$ can be obtained in the same way as in the previous case. Applying Hölder's inequality we obtain

$$\begin{aligned}
|U_t|^\alpha &\leq \left(\int_{0+}^t \int_{\mathbb{R}} \mathcal{E}(R)_{s-}^{-1/\alpha}\mathcal{E}(R)_{s-}^{1/\alpha-1}|x|\mathbf{1}_{\{|x|>1\}}\mu_X(ds,dx)\right)^\alpha \\
&\leq \left(\int_{0+}^t \int_{\mathbb{R}} \mathcal{E}(R)_{s-}^{-1}|x|^\alpha\mathbf{1}_{\{|x|>1\}}\mu_X(ds,dx)\right) \\
&\quad\times \left(\int_{0+}^t \int_{\mathbb{R}} \mathcal{E}(R)_{s-}^{-1}\mathbf{1}_{\{|x|>1\}}\mu_X(ds,dx)\right)^{\alpha-1} \\
&\leq \left(\int_{0+}^t \int_{\mathbb{R}} \mathcal{E}(R)_{s-}^{-1}|x|^\alpha\mathbf{1}_{\{|x|>1\}}\mu_X(ds,dx)\right)^\alpha.
\end{aligned}$$

Then, using Markov's inequality and the compensation formula, we obtain

$$\begin{aligned}
\mathbf{P}\left(\sup_{0\leq t\leq T}|U_t| \geq \frac{y}{4}\right) &\leq \frac{4^\alpha}{y^\alpha}\mathbf{E}\left(\sup_{0\leq t\leq T}|U_t|^\alpha\right) \\
&= \left(\int_{\mathbb{R}} |x|^\alpha\mathbf{1}_{\{|x|>1\}}K_X(dx)\right)^\alpha \mathbf{E}(I_T^\alpha).
\end{aligned}$$

This finishes the proof in the case $1 < \alpha \leq 2$.

Finally, suppose that $\alpha \geq 2$. The estimation for $\mathbf{P}\left(\sup_{0\leq t\leq T}|U_t| \geq \frac{y}{4}\right)$ still works in this case. Moreover, since $\mathcal{E}(R)_{t-}^{-1}x\mathbf{1}_{\{|x|\leq 1\}} \in F_2$, we obtain, applying part (b) of Proposition 3.2.5 that

$$\begin{aligned}
\mathbf{P}\left(\sup_{0\leq t\leq T}|M_t^d| \geq \frac{y}{4}\right) \leq& K_2\mathbf{E}\left[\left(\int_0^T \int_{\mathbb{R}} \mathcal{E}(R)_{s-}^{-2}x^2\mathbf{1}_{\{|x|\leq 1\}}K_X(dx)ds\right)^{\alpha/2}\right] \\
&+ K_3\mathbf{E}\left(\int_0^T \int_{\mathbb{R}} \mathcal{E}(R)_{s-}^{-\alpha}|x|^\alpha\mathbf{1}_{\{|x|\leq 1\}}K_X(dx)ds\right) \\
=& K_2\left(\int_{\mathbb{R}} x^2\mathbf{1}_{\{|x|\leq 1\}}K_X(dx)\right)^{\alpha/2}\mathbf{E}(J_T^{\alpha/2}) \\
&+ K_3\left(\int_{\mathbb{R}} |x|^\alpha\mathbf{1}_{\{|x|\leq 1\}}K_X(dx)\right)\mathbf{E}(J_T(\alpha)).
\end{aligned}$$

Note that the right-hand side is finite since $|x|^\alpha\mathbf{1}_{\{|x|\leq 1\}} \leq |x|^2\mathbf{1}_{\{|x|\leq 1\}}$ when $\alpha \geq 2$. This finishes the proof of (3.7). Then we take ln from both sides of (3.7), we divide the inequality by $\ln(y)$ and we take $\lim_{y\to+\infty}$, and, then $\lim_{\alpha\to\beta_T}$ to get (3.8). $\square$

## 3.3 Lower Bound and Logarithmic Asymptotic

The natural question is then whether the bound stated in Theorem 3.2.1 is optimal. In this section, we partially answer this question by giving, under some simple additional conditions on the Lévy triplet of $X$, a lower bound for the ruin probability and, from this, deducing that the bound is at least optimal for a large class of processes $X$ when considering the logarithmic asymptotic. The main result of this section is the following.

**Theorem 3.3.1.** *Let $T > 0$. Assume that for $\gamma_T \geq 1$ we have $\mathbf{E}(I_T^{\gamma_T}) = \infty$. Additionally, assume that*

$$\int_{\{|x|>1\}} |x| K_X(dx) < \infty \ \ (\text{or equivalently that } \mathbf{E}(|X_1|) < \infty) \tag{3.11}$$

*and that*

$$\mathbf{E}(X_1) = a_X + \int_{\{|x|>1\}} x K_X(dx) < 0 \text{ or } \sigma_X > 0. \tag{3.12}$$

*Then, for all $\delta > 0$, there exists a positive numerical sequence $(y_n)_{n\in\mathbb{N}}$ increasing to $\infty$ such that, for all $C > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,*

$$\mathbf{P}(\tau(y_n) \leq T) \geq \frac{C}{y_n^{\gamma_T} \ln(y_n)^{1+\delta}}.$$

*Moreover,*

$$\limsup_{y\to\infty} \frac{\ln\left(\mathbf{P}(\tau(y) \leq T)\right)}{\ln(y)} \geq -\gamma_T.$$

Thus, under the conditions of Theorems 3.2.1 and 3.3.1 with $\gamma_T = \beta_T$, we obtain immediately the logarithmic asymptotic for the ruin probability

$$\limsup_{y\to\infty} \frac{\ln\left(\mathbf{P}(\tau(y) \leq T)\right)}{\ln(y)} = -\beta_T.$$

Concerning the lower bound for the ultimate ruin probability, we will prove the following.

**Corollary 3.3.2.** *Assume that* $\mathbf{E}(I_\infty) < \infty$ *and* $\mathbf{E}(J_\infty) < \infty$ *and that there exists* $\gamma_\infty > 1$ *such that* $\mathbf{E}(I_\infty^{\gamma_\infty}) = \infty$ *and* $\mathbf{E}(J_\infty^{\gamma_\infty/2}) = \infty$*. Assume that* $X$ *verifies* (3.11) *and* (3.12)*. Then,*

$$\limsup_{y\to\infty} \frac{\ln\left(\mathbf{P}(\tau(y) < \infty)\right)}{\ln(y)} \geq -\gamma_\infty.$$

Again, under the assumptions of the Corollaries 3.2.2 and 3.3.2 with $\gamma_\infty = \beta_\infty$ we can obtain the logarithmic asymptotic for the ultimate ruin probability

$$\limsup_{y\to\infty} \frac{\ln\left(\mathbf{P}(\tau(y) < \infty)\right)}{\ln(y)} = -\beta_\infty.$$

We introduce the notation $x^{+,p} = (\max(x,0))^p$, for all $x \in \mathbb{R}$ and $p > 0$, and give three simple preliminary results which will allow to reduce the problem of finding a lower bound for the ruin probability to the problem of proving that a certain expectation is infinite.

**Lemma 3.3.3.** *Suppose that a random variable* $Z \geq 0$ ($\mathbf{P} - a.s.$) *satisfies* $\mathbf{E}(Z^p) = \infty$*, for some* $p > 0$*. Then, for all* $\delta > 0$*, there exists a positive numerical sequence* $(y_n)_{n\in\mathbb{N}}$ *increasing to* $\infty$ *such that, for all* $C > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$*,*

$$\mathbf{P}(Z > y_n) \geq \frac{C}{y_n^p \ln(y_n)^{1+\delta}}.$$

*Proof.* If $Z \geq 0$ ($\mathbf{P} - a.s.$) is a random variable and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a function of class $C^1$ with positive derivative, then, using Fubini's theorem, we obtain

$$g(0) + \int_0^\infty g'(u)\mathbf{P}(Z > u)du = g(0) + \mathbf{E}\left(\int_0^Z g'(u)du\right) = \mathbf{E}(g(Z)).$$

Applying this to the function $g(z) = z^p$ with $p > 0$ we obtain, for all $y > 1$,

$$\int_y^\infty u^{p-1}\mathbf{P}(Z > u)du = \infty.$$

Moreover, for all $\delta > 0$,

$$\sup_{u \geq y}[u^p \ln(u)^{1+\delta}\mathbf{P}(Z > u)] \int_y^\infty \frac{du}{u\ln(u)^{1+\delta}} \geq \int_y^\infty u^{p-1}\mathbf{P}(Z > u)du.$$

So, since $\int_y^\infty \frac{du}{u\ln(u)^{1+\delta}} < \infty$, we obtain, for all $y > 1$,

$$\sup_{u \geq y}[u^p \ln(u)^{1+\delta}\mathbf{P}(Z > u)] = \infty.$$

Therefore, there exists a numerical sequence $(y_n)_{n\in\mathbb{N}}$ increasing to $\infty$ such that,

$$\lim_{n\to\infty} y_n^p \ln(y_n)^{1+\delta}\mathbf{P}(Z > y_n) = \infty.$$

□

**Lemma 3.3.4.** *Assume that $X$ and $Y$ are independent random variables with $\mathbf{E}(Y) = 0$. Assume that $p \geq 1$. Then, $\mathbf{E}[X^{+,p}] \leq \mathbf{E}[(X+Y)^{+,p}]$.*

*Proof.* For each $x \in \mathbb{R}$, we define the function $h_x : y \mapsto (x+y)^{+,p}$ on $\mathbb{R}$. Since $p \geq 1$, $h_x$ is a convex function and we obtain, using Jensen's inequality, that for each $x \in \mathbb{R}$,

$$\mathbf{E}[(x+Y)^{+,p}] = \mathbf{E}[h_x(Y)] \geq h_x(\mathbf{E}(Y)) = h_x(0) = x^{+,p}.$$

We obtain the desired result by integrating w.r.t. the law of $X$. □

**Lemma 3.3.5.** *Let $T > 0$. Assume that $a < 0$ or $\sigma > 0$ and that there exists $\gamma > 0$ such that $\mathbf{E}(I_T^\gamma) = \infty$. Then, $\mathbf{E}[(-aI_T - \sigma W_{J_T})^{+,\gamma}] = \infty$.*

*Proof.* Suppose first that $a < 0$ and $\sigma = 0$. Then,

$$\mathbf{E}[(-aI_T - \sigma W_{J_T})^{+,\gamma}] = |a|^\gamma \mathbf{E}(I_T^\gamma) = \infty.$$

Next, suppose that $a \leq 0$ and $\sigma > 0$. In that case, using the identities in law $W \stackrel{d}{=} -W$ and $W_{J_T} \stackrel{d}{=} \sqrt{J_T}W_1$, the Cauchy-Schwarz inequality and the independence between $W_1$ and $J_T$, we obtain

$$\begin{aligned}\mathbf{E}[(-aI_T - \sigma W_{J_T})^{+,\gamma}] &\geq \mathbf{E}[(\sigma\sqrt{J_T}W_1)^{+,\gamma}] = \sigma^\gamma \mathbf{E}(W_1^{+,\gamma})\mathbf{E}(J_T^{\gamma/2}) \\ &\geq \sigma^\gamma \mathbf{E}(W_1^{+,\gamma})T^{-\gamma/2}\mathbf{E}(I_T^\gamma) = \infty.\end{aligned}$$

Finally, if $a > 0$ and $\sigma > 0$, using the fact that $W \overset{d}{=} -W$, that $W_{J_T} \overset{d}{=} \sqrt{J_T} W_1$ and choosing $C > 1$, we obtain that

$$\begin{aligned}
\mathbf{E}[(-aI_T - \sigma W_{J_T})^{+,\gamma}] &= \mathbf{E}[(-|a|I_T + \sigma\sqrt{J_T}W_1)^{+,\gamma}] \\
&\geq \mathbf{E}[(-|a|I_T + \sigma\sqrt{J_T}W_1)^{+,\gamma}\mathbf{1}_{\{\sigma\sqrt{J_T}W_1 \geq C|a|I_T\}}] \\
&\geq \mathbf{E}[((C-1)|a|I_T)^{\gamma}\mathbf{1}_{\{\sigma\sqrt{J_T}W_1 \geq C|a|I_T\}}].
\end{aligned}$$

Since $\frac{I_T}{\sqrt{J_T}} \leq \sqrt{T}$, by Cauchy-Schwarz's inequality, we obtain using the independence between $W_1$ and $I_T$

$$\begin{aligned}
\mathbf{E}[(-aI_T - \sigma W_{J_T})^{+,\gamma}] &\geq \mathbf{E}\left[((C-1)|a|I_T)^{\gamma}\mathbf{1}_{\left\{W_1 \geq \frac{C|a|\sqrt{T}}{\sigma}\right\}}\right] \\
&= \mathbf{P}\left(W_1 \geq \frac{C|a|\sqrt{T}}{\sigma}\right)(C-1)^{\gamma}|a|^{\gamma}\mathbf{E}(I_T^{\gamma}) = \infty.
\end{aligned}$$

□

We can now give the proof of Theorem 3.3.1.

*Proof of Theorem 3.3.1.* The assumptions imply $\int_{|x|>1} |x| K_X(dx) < \infty$ and so, using Equation (3.3) and Proposition 3.2.4, we obtain

$$\begin{aligned}
\mathbf{P}(\tau(y) \leq T) &\geq \mathbf{P}\left(\sup_{0\leq t\leq T}\left(-\int_{0+}^{t}\frac{dX_s}{\mathcal{E}(R)_{s-}}\right) > y\right) \\
&\geq \mathbf{P}((-\delta_X I_T - \sigma_X W_{J_T} - N_T^d)^+ > y),
\end{aligned}$$

where $\delta_X$ and $N^d = (N_t^d)_{t\in[0,T]}$ are defined as in Proposition 3.2.4.

Then, by independence, we get

$$\begin{aligned}
&\mathbf{E}[(-\delta_X I_T - \sigma_X W_{J_T} - N_T^d)^{+,\gamma_T}] \\
&\quad = \int_{\mathbb{D}_T} \mathbf{E}[(-\delta_X I_T(q) - \sigma_X W_{J_T(q)} - N_T^d(q))^{+,\gamma_T}]\mathbf{P}_{\mathcal{E}(R)}(dq),
\end{aligned}$$

where $\mathbb{D}_T$ is the Skorokhod space of càdlàg functions on $[0, T]$, the measure $\mathbf{P}_{\mathcal{E}(R)}$ is the law of $(\mathcal{E}(R)_t)_{t\in[0,T]}$, $I_T(q) = \int_0^T \frac{ds}{q_s}$, $J_T(q) = \int_0^T \frac{ds}{q_s^2}$ and

$$\begin{aligned}
N_T^d(q) = &\int_{0+}^{T}\int_{\{|x|\leq 1\}}\frac{x}{q_{s-}}(\mu_X(ds,dx) - K_X(dx)ds) \\
&+ \int_{0+}^{T}\int_{\{|x|>1\}}\frac{x}{q_{s-}}(\mu_X(ds,dx) - K_X(dx)ds).
\end{aligned}$$

Denote by $N'_T(q)$ and $N''_T(q)$ the two terms on the r.h.s. of the equation above. Fixing $q \in \mathbb{D}_T$, we now prove that $\mathbf{E}(N'_T(q)) = 0$ and $\mathbf{E}(N''_T(q)) = 0$. First, note that by Theorem 1 p.176 in [Liptser and Shiryayev, 1989] and the compensation formula, we find that

$$\begin{aligned}
\mathbf{E}([N'_\cdot(q), N'_\cdot(q)]_T) &= \mathbf{E}\left(\int_{0+}^T \int_{\{|x|\le 1\}} \frac{x^2}{q_{s-}^2} \mu_X(ds,dx)\right) \\
&= \mathbf{E}\left(\int_0^T \int_{\{|x|\le 1\}} \frac{x^2}{q_s^2} K_X(dx)ds\right) \\
&= \left(\int_0^T \frac{ds}{q_s^2}\right)\left(\int_{\{|x|\le 1\}} x^2 K_X(dx)\right).
\end{aligned}$$

Then, since $q$ a strictly positive càdlàg function on a compact interval, the integral $\int_0^T \frac{ds}{q_s^2} < \infty$ and since $\int_{\{|x|\le 1\}} x^2 K_X(dx) < \infty$ by definition of the Lévy measure, we have $\mathbf{E}([N'_\cdot(q), N'_\cdot(q)]_T) < \infty$. This shows that $N'(q)$ is a (locally square-integrable) martingale and so $\mathbf{E}(N'_T(q)) = 0$. For the second term, similarly we have

$$\int_0^T \int_{\{|x|>1\}} \frac{|x|}{q_s} K_X(dx)ds = \left(\int_0^T \frac{ds}{q_s}\right)\left(\int_{\{|x|>1\}} |x| K_X(dx)\right) < \infty.$$

Therefore, by Proposition II.1.28 p.72 in [Jacod and Shiryaev, 2003] and the compensation formula, we have

$$\begin{aligned}
\mathbf{E}(N''_T(q)) = \mathbf{E}&\left(\int_{0+}^T \int_{|x|>1} \frac{x}{q_{s-}} \mu_X(ds,dx)\right) \\
&- \mathbf{E}\left(\int_0^T \int_{|x|>1} \frac{x}{q_{s-}} K_X(dx)ds\right) = 0.
\end{aligned}$$

Now, since the random variables $-\delta_X I_T(q) - \sigma_X W_{J_T(q)}$ and $-N_T^d(q)$ are independent and $\mathbf{E}(N_T^d(q)) = 0$, for all $q \in \mathbb{D}_T$, we can apply Lemma 3.3.4 to obtain

$$\mathbf{E}[(-\delta_X I_T - \sigma_X W_{J_T} - N_T^d)^{+,\gamma_T}] \ge \mathbf{E}[(-\delta_X I_T - \sigma_X W_{J_T})^{+,\gamma_T}].$$

Then, using Lemma 3.3.3 and Lemma 3.3.5 with $a = \delta_X$, $\sigma = \sigma_X$, the variable $Z = (-\delta_X I_T - \sigma_X W_{J_T})^+$ and $p = \gamma_T$, we can conclude that for all $\delta > 0$, there exists a strictly positive sequence $(y_n)_{n\in\mathbb{N}}$ increasing to $\infty$ such that, for all $C > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$\mathbf{P}(\tau(y_n) \leq T) \geq \frac{C}{y_n^{\gamma_T} \ln(y_n)^{1+\delta}}.$$

The above implies that

$$\limsup_{y\to\infty} \frac{\ln\left(\mathbf{P}(\tau(y) \leq T)\right)}{\ln(y)} \geq -\gamma_T + \lim_{n\to\infty} \frac{\ln(C) - \ln(\ln(y_n)^{1+\delta})}{\ln(y_n)} = -\gamma_T$$

and this finishes the proof. □

We can use a similar method to prove Corollary 3.3.2.

*Proof of Corollary 3.3.2.* First of all we show that the process $N^d$ appearing in the proof of Theorem 3.3.1 is uniformly integrable. We take first

$$N'_t = \int_{0+}^t \int_{\{|x|<1\}} \frac{x}{\mathcal{E}(R)_{s-}} (\mu_X(ds, dx) - K_X(dx)ds)$$

From the proof of Theorem 3.3.1, we see that

$$\begin{aligned} \sup_{t\geq 0} \mathbf{E}[(N'_t)^2] &= \sup_{t\geq 0} \mathbf{E}\left(\int_0^t \int_{\{|x|<1\}} \frac{x^2}{\mathcal{E}^2(R)_{s-}} K_X(dx)ds\right) \\ &= \mathbf{E}(J_\infty) \int_{\mathbb{R}} x^2 \mathbf{1}_{\{|x|<1\}} K_X(dx) < \infty \end{aligned}$$

the process $N'$ is uniformly integrable, by de la Vallée-Poussin's criterion.

Now, let

$$N''_t = \int_{0+}^t \int_{\{|x|>1\}} \frac{x}{\mathcal{E}(R)_{s-}} \mu_X(ds, dx) - \int_0^t \int_{\{|x|>1\}} \frac{x}{\mathcal{E}(R)_{s-}} K_X(dx)ds$$

By the compensation formula

$$\mathbf{E}(|N''_\infty|) \leq 2\mathbf{E}(I_\infty) \int_{\{|x|>1\}} |x| \, K_X(dx) < \infty$$

and this shows that $N''$ has a finite $(\mathbf{P}-a.s.)$ limit as $t \to \infty$. Hence, $N^d$ is uniformly integrable and $\mathbf{E}(N^d_\infty) = 0$. From Proposition 3.2.4 we get that

$$\int_{0+}^{\infty} \frac{dX_s}{\mathcal{E}(R)_{s-}} \stackrel{d}{=} \delta_X I_\infty + \sigma_X W_{J_\infty} + N^d_\infty.$$

And thus, imitating the proof of Lemma 3.3.5, we conclude that

$$\mathbf{E}[(-\delta_X I_\infty - \sigma_X W_{J_\infty})^{+,\gamma_\infty}] = \infty.$$

Finally, from Lemma 3.3.3 with $Z = (-\delta_X I_\infty - \sigma_X W_{J_\infty})^+$ and $p = \gamma_\infty$ we obtain the claimed result. $\square$

## 3.4 Moments of Exponential Functionals of Lévy Processes

While Theorems 3.2.1 and 3.3.1 give the qualitative features of the ruin probabilities, their usefulness rely on the knowledge of $\beta_T$ and $\beta_\infty$ whose expression depend on the moments (and ultimately the law) of $I_T$ and $I_\infty$. In this section, we briefly point to some known results on the law of these exponential functionals and give a method to compute $\beta_T$ and $\beta_\infty$ in the case when $R$ is a Lévy process. We also apply these results to some classic models in mathematical finance.

The question of the existence of the moments of $I_\infty$ and the formula in the case when $R$ is a subordinator were considered in [Bertoin and Yor, 2005], [Carmona et al., 1997] and [Salminen and Vostrikova, 2018]. In the case when $R$ is a Lévy process, the question of the existence of the density of the law of $I_\infty$, PDE equations for the density and the asymptotics for the law were investigated in [Behme, 2015], [Behme and Lindner, 2015], [Bertoin et al., 2008], [Dufresne, 1990], [Erickson and Maller, 2005], [Gjessing and Paulsen, 1997], [Kuznetsov et al., 2012], [Pardo et al., 2013], [Patie and Savov, 2018] and [Rivero, 2012]. In the more general case of processes with independent increments, conditions for the existence of the moments and recurrent equations for the moments were studied in [Salminen and Vostrikova, 2018] and [Salminen and Vostrikova, 2019]. The existence of the density of such functionals and the corresponding PDE equations were considered in [Vostrikova, 2018].

Recall that when $R$ is a Lévy process with characteristics $(a_R, \sigma_R^2, K_R)$, its exponential transform $\hat{R}$ is also a Lévy process whose characteristics we denote $(a_{\hat{R}}, \sigma_{\hat{R}}^2, K_{\hat{R}})$. Recall also that the jumps of the processes are related by the equation $\Delta \hat{R}_t = \ln(1 + \Delta R_t)$, for all $t \geq 0$. The first result concerns $\beta_T$.

**Proposition 3.4.1.** *Suppose that $R$ is a Lévy process. For $\alpha > 0$ and $T > 0$ the following conditions are equivalent:*

*(i)* $\mathbf{E}(J_T(\alpha)) < \infty$,

*(ii)* $\int_{\{|x|>1\}} e^{-\alpha x} K_{\hat{R}}(dx) < \infty$,

*(iii)* $\int_{-1}^{\infty} \mathbf{1}_{\{|\ln(1+x)|>1\}} (1+x)^{-\alpha} K_R(dx) < \infty$.

*Proof.* By Fubini's theorem, we obtain

$$\mathbf{E}(J_T(\alpha)) = \mathbf{E}\left(\int_0^T e^{-\alpha \hat{R}_t} dt\right) = \int_0^T \mathbf{E}(e^{-\alpha \hat{R}_t}) dt.$$

So, $\mathbf{E}(J_T(\alpha)) < \infty$ is equivalent to $\mathbf{E}(e^{-\alpha \hat{R}_t}) < \infty$, for all $t \geq 0$, which, by Theorem 25.3, p.159 in [Sato, 1999], is equivalent to

$$\int_{\{|x|>1\}} e^{-\alpha x} K_{\hat{R}}(dx) < \infty.$$

Then, note that

$$\begin{aligned}
\int_{\{|x|>1\}} e^{-\alpha x} K_{\hat{R}}(dx) &= \int_0^1 \int_{\{|x|>1\}} e^{-\alpha x} K_{\hat{R}}(dx) ds \\
&= \mathbf{E}\left(\sum_{0<s\leq 1} e^{-\alpha \Delta \hat{R}_s} \mathbf{1}_{\{|\Delta \hat{R}_s|>1\}}\right) \\
&= \mathbf{E}\left(\sum_{0<s\leq 1} (1+\Delta R_s)^{-\alpha} \mathbf{1}_{\{|\ln(1+\Delta R_s)|>1\}}\right) \\
&= \int_{-1}^{\infty} \mathbf{1}_{\{|\ln(1+x)|>1\}} (1+x)^{-\alpha} K_R(dx).
\end{aligned}$$

$\square$

This proposition allows us to compute $\beta_T$ in two standard models of mathematical finance.

**Example 3.4.2** (Jump-diffusion log-returns)**.** Suppose that $\hat{R}$ is given by $\hat{R}_t = a_{\hat{R}}t + \sigma_{\hat{R}}W_t + \sum_{n=0}^{N_t} Y_n$, where $a_{\hat{R}} \in \mathbb{R}$, $\sigma_{\hat{R}} \geq 0$, $W = (W_t)_{t\geq 0}$ is a standard Brownian motion and $N = (N_t)_{t\geq 0}$ is a Poisson process with rate $\gamma > 0$, and $(Y_n)_{n\in\mathbb{N}}$ is a sequence of iid random variables. Suppose, in addition, that all processes involved are independent. If for $(Y_n)_{n\in\mathbb{N}}$ we take any sequence of i.i.d. random variables with $\mathbf{E}(e^{-\alpha Y_1}) < \infty$, for all $\alpha > 0$, then $\beta_T = +\infty$. If for $(Y_n)_{n\in\mathbb{N}}$ we take a sequence of i.i.d. random variables with $\mathbf{E}(e^{-\alpha Y_1}) < \infty$, when $\alpha < \alpha_0$, for some $\alpha_0 > 0$, and $\mathbf{E}(e^{-\alpha_0 Y_1}) = +\infty$, then $\beta_T = \alpha_0$.

**Example 3.4.3** (Tempered stable log-returns)**.** Suppose that $\hat{R}$ is a Lévy process with triplet $(a_{\hat{R}}, \sigma^2_{\hat{R}}, K_{\hat{R}})$, where $a_{\hat{R}} \in \mathbb{R}$, $\sigma_{\hat{R}} \geq 0$ and $K_{\hat{R}}$ is the measure on $\mathbb{R}$ given by

$$K_{\hat{R}}(dx) = \left(C_1|x|^{-(1+\alpha_1)}e^{-\lambda_1|x|}\mathbf{1}_{\{x<0\}} + C_2x^{-(1+\alpha_2)}e^{-\lambda_2 x}\mathbf{1}_{\{x>0\}}\right)dx,$$

where $C_1, C_2 > 0$, $\lambda_1, \lambda_2 > 0$ and $0 < \alpha_1, \alpha_2 < 2$. This specification includes as special cases the Kou, CGMY and variance-gamma models (see e.g. Section 4.5 p.119 in [Cont and Tankov, 2004]). We will show that $\beta_T = \lambda_1$. Note that, using Proposition 3.4.1 and the change of variables $y = -x$, we see that $\mathbf{E}(J_T(\alpha)) < \infty$, for $\alpha > 0$, is equivalent to

$$C_1 \int_1^\infty y^{-(1+\alpha_1)}e^{-(\lambda_1-\alpha)y}dy + C_2\int_1^\infty x^{-(1+\alpha_2)}e^{-(\alpha+\lambda_2)x}dx < \infty.$$

But, the first integral converges if $\alpha \leq \lambda_1$ and diverges if $\alpha > \lambda_1$ and second integral always converges. Now, if $\alpha \geq 2$, it is easy to show that $\mathbf{E}(J_T(\alpha)) < \infty$ implies $\mathbf{E}(J_T^{\alpha/2}) < \infty$ (see Lemma 3.2.3). Thus, if $\lambda_1 \geq 2$, we have $\beta_T = \lambda_1$.

The method to compute $\beta_\infty$ is even more explicit since it can be computed from the Laplace exponent as proven in the following proposition.

**Proposition 3.4.4.** *Suppose that $R$ is a Lévy process and that $\hat{R}$ admits a Laplace transform, for all $t \geq 0$, i.e. for $\alpha > 0$*

$$\mathbf{E}(\exp(-\alpha\hat{R}_t)) = \exp(t\psi_{\hat{R}}(\alpha))$$

*and that its Laplace exponent $\psi_{\hat{R}}$ has a strictly positive root $\beta_0$. Then, the following conditions are equivalent:*

*(i)* $\mathbf{E}(I_\infty^\alpha) < \infty$,

*(ii)* $\mathbf{E}(J_\infty^{\alpha/2}) < \infty$,

*(iii)* $\mathbf{E}(J_\infty(\alpha)) < \infty$,

*(iv)* $\alpha < \beta_0$.

*Therefore, $\beta_\infty = \beta_0$.*

*Proof.* Note that, for any $\alpha > 0$ and $k > 0$,

$$\begin{aligned}\exp(t\psi_{\hat{R}}(\alpha)) = \mathbf{E}(\exp(-\alpha\hat{R}_t)) &= \mathbf{E}\left(\exp\left(-\frac{\alpha}{k}k\hat{R}_t\right)\right)\\ &= \exp\left(t\psi_{k\hat{R}}\left(\frac{\alpha}{k}\right)\right).\end{aligned}$$

Therefore, $\psi_{\hat{R}}(\alpha) = \psi_{k\hat{R}}\left(\frac{\alpha}{k}\right)$, for all $\alpha > 0$ and $k > 0$. Then, Lemma 3 in [Rivero, 2012] yields the desired result. □

**Remark 3.4.5.** The root of the Laplace exponent was already identified as the relevant quantity for the tails of $\mathbf{P}(\tau(y) < \infty)$ in [Paulsen, 2002].

Thus, we can compute $\beta_\infty$ in two important examples.

**Example 3.4.6** (Black-Scholes log-returns)**.** Suppose that $R_t = a_R t + \sigma_R W_t$, for all $t \geq 0$, where $a_R \in \mathbb{R}$, $\sigma_R > 0$ and $W = (W_t)_{t\geq 0}$ is a standard Brownian motion, then $\hat{R}_t = \left(a_R - \frac{\sigma_R^2}{2}\right)t + \sigma_R W_t$, for all $t \geq 0$. Thus, we obtain $\psi_{\hat{R}}(\alpha) = -\left(a_R - \frac{1}{2}\sigma_R^2\right)\alpha + \frac{\sigma_R^2}{2}\alpha^2$ and, by Proposition 3.4.4, $\beta_\infty = \frac{2a_R}{\sigma_R^2} - 1$. We remark that this coincides with the results in e.g. [Frolova et al., 2002] and [Kabanov and Pergamentshchikov, 2016].

**Example 3.4.7** (Jump-diffusion log-returns)**.** Suppose that $\hat{R}_t = a_{\hat{R}}t + \sigma_{\hat{R}}W_t + \sum_{n=0}^{N_t} Y_n$, where $a_{\hat{R}} \in \mathbb{R}$, $\sigma_{\hat{R}} \geq 0$ and $W = (W_t)_{t\geq 0}$ is a standard Brownian motion and $N = (N_t)_{t\geq 0}$ is a Poisson process with rate $\gamma > 0$, and $(Y_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. random variables with $\mathbf{E}(e^{-\alpha Y_1}) < \infty$, for all

$\alpha > 0$. Suppose, in addition, that all processes involved are independent. It is easy to see that, for all $\alpha > 0$,

$$\psi_{\hat{R}}(\alpha) = -a_{\hat{R}}\alpha + \frac{\sigma_{\hat{R}}^2}{2}\alpha^2 + \gamma\left(\mathbf{E}(e^{-\alpha Y_1}) - 1\right).$$

Now, it is possible to show (see e.g. [Spielmann, 2018]) that the equation $\psi_{\hat{R}}(\alpha) = 0$ has an unique non-zero solution if, and only if, $\hat{R}$ is not a subordinator and $\psi'(0+) < 0$ which, under some additional conditions to invert the differentiation and expectation operators, is equivalent to $a_{\hat{R}} > \gamma\mathbf{E}(Y_1)$. In that case, $\beta_\infty$ is the unique non-zero real solution of this equation.

## 3.5 Conditions for Ruin with Probability One

To complete our study of the ruin problem for the GOU process we give, in this section, sufficient conditions for ruin with probability one when $X$ has positive jumps bounded by $a > 0$ and verifies one of the following conditions

$$a_X < 0 \text{ or } \sigma_X > 0 \text{ or } K_X([-a, a]) > 0. \tag{3.13}$$

**Theorem 3.5.1.** *Assume that $X$ verifies the condition (3.13). In addition assume that $(\mathbf{P} - a.s.)$, $I_\infty = \infty$, $J_\infty = \infty$ and that there exists a limit*

$$\lim_{t\to\infty} \frac{I_t}{\sqrt{J_t}} = L$$

*with $0 < L < \infty$. Then, for all $y > 0$,*

$$\mathbf{P}(\tau(y) < \infty) = 1.$$

In the case of Lévy processes we express the conditions on the exponential functionals in terms of their characteristics to get the following result which is similar to the known results in e.g. [Kabanov and Pergamentshchikov, 2018] and [Paulsen, 1998].

**Corollary 3.5.2.** *Assume that $X$ verifies the condition (3.13). Suppose that $R$ is a Lévy process with characteristic triplet $(a_R, \sigma_R^2, K_R)$ satisfying*

$$\int_{-1}^{\infty} |\ln(1+x)| \mathbf{1}_{\{|\ln(1+x)|>1\}} K_R(dx) < \infty \tag{3.14}$$

*and*

$$a_R - \frac{\sigma_R^2}{2} + \int_{-1}^{\infty} (\ln(1+x) - x\mathbf{1}_{\{|\ln(1+x)|\leq 1\}}) K_R(dx) < 0.$$

*Then, for all $y > 0$,*

$$\mathbf{P}(\tau(y) < \infty) = 1.$$

*Proof of Theorem 3.5.1.* We have, for all $y > 0$,

$$\begin{aligned}
\mathbf{P}(\tau(y) < \infty) &\geq \mathbf{P}\left(\sup_{t\geq 0}\left(-\int_{0+}^{t} \frac{dX_s}{\mathcal{E}(R)_{s-}}\right) > y\right) \\
&\geq \mathbf{P}\left(\limsup_{t\to\infty}\left(-\int_{0+}^{t} \frac{dX_s}{\mathcal{E}(R)_{s-}}\right) > y\right) \\
&\geq \mathbf{P}\left(\limsup_{t\to\infty}\left(-\int_{0+}^{t} \frac{dX_s}{\mathcal{E}(R)_{s-}}\right) = +\infty\right).
\end{aligned}$$

But, the independence of $X$ and $R$ implies

$$\begin{aligned}
&\mathbf{P}\left(\limsup_{t\to\infty}\left(-\int_{0+}^{t} \frac{dX_s}{\mathcal{E}(R)_{s-}}\right) = \infty\right) \\
&\qquad = \int_{\mathbb{D}} \mathbf{P}\left(\limsup_{t\to\infty}\left(-\int_{0+}^{t} \frac{dX_s}{q_{s-}}\right) = \infty\right) \mathbf{P}_{\mathcal{E}(R)}(dq)
\end{aligned}$$

where $\mathbb{D}$ is Skorokhod space of càdlàg functions on $\mathbb{R}_+$. We remark that the event

$$\left\{\limsup_{t\to\infty}\left(-\int_{0+}^{t} \frac{dX_s}{q_{s-}}\right) = \infty\right\}$$

is a tail event for an additive process $\left(-\int_{0+}^{t} \frac{dX_s}{q_{s-}}\right)_{t\geq 0}$ and this event has either probability 0 or 1. We will now show that

$$\mathbf{P}\left(\limsup_{t\to\infty}\left(-\int_{0+}^{t} \frac{dX_s}{q_{s-}}\right) = \infty\right) = 1$$

on the set

$$Q = \left\{ q \in \mathbb{D} : I_\infty(q) = \infty, J_\infty(q) = \infty, \lim_{t\to\infty} \frac{I_t(q)}{\sqrt{J_t(q)}} = L(q),\ 0 < L(q) < \infty \right\}$$

of probability 1. Here we denote as previously $I_t(q) = \int_0^t q_s^{-1} ds$ and $J_t(q) = \int_0^t q_s^{-2} ds$.

Imitating the proof of Proposition 3.2.4 for the truncation function $\mathbf{1}_{\{|x|\le a\}}$, we obtain

$$\begin{aligned}
&\mathbf{P}\left(\limsup_{t\to\infty}\left(-\int_{0+}^t \frac{dX_s}{q_{s-}}\right) = \infty\right)\\
&= \mathbf{P}\left(\limsup_{t\to\infty}\left(-a_X I_t(q) - \sigma_X W_{J_t(q)} - M_t^{a,d}(q) - U_t^a(q)\right) = \infty\right)\\
&\geq \mathbf{P}\left(\limsup_{t\to\infty}\left(-a_X I_t(q) - \sigma_X W_{J_t(q)} - M_t^{a,d}(q)\right) = \infty\right),
\end{aligned}$$

where

$$M_t^{a,d}(q) = \int_{0+}^t \int_{\mathbb{R}} x\mathbf{1}_{\{|x|\le a\}}(\mu_X(ds,dx) - K_X(dx)ds)$$

and

$$U_t^a(q) = \int_{0+}^t \int_{\mathbb{R}} x\mathbf{1}_{\{|x|> a\}}\mu_X(ds,dx).$$

The last inequality follows from the assumption $K_X(]a,+\infty[) = 0$ which implies that $U_t^a \le 0$ for all $t \ge 0$.

Next, $H_t(q) = -\sigma_X W_{J_t(q)} - M_t^{a,d}(q)$ is a locally square-integrable martingale and using the independence of the terms in the Lévy-Itô decomposition of $X$, we can obtain its variance :

$$\mathbf{E}(H_t(q)^2) = \left(\sigma_X^2 + \int_{\mathbb{R}} x^2 \mathbf{1}_{\{|x|\le a\}} K_X(dx)\right) J_t(q) = \sigma_\xi^2 J_t(q)$$

with $\sigma_\xi^2 = \sigma_X^2 + \int_{\mathbb{R}} x^2 \mathbf{1}_{\{|x|\le a\}} K_X(dx)$.

Now if $\sigma_\xi = 0$, then by assumption we would have $a_X < 0$ and $M_t^{a,d} = 0$, for all $t \ge 0$, and thus

$$\mathbf{P}\left(\limsup_{t\to\infty}\left(-\int_{0+}^t \frac{dX_s}{q_{s-}}\right) = \infty\right) \ge \mathbf{P}\left(\limsup_{t\to\infty}\left(-a_X I_t(q)\right) = \infty\right) = 1$$

since $I_\infty(q) = \infty$ on the set $Q$.

If $\sigma_\xi > 0$, we take an increasing sequence $(t_n)_{n\in\mathbb{N}}$ starting from zero and increasing to $\infty$ and, for all $n \in \mathbb{N}^*$ and $0 \le k \le n$ we set

$$\tilde{H}_{n,k} = \frac{H_{t_k}(q) - H_{t_{k-1}}(q)}{\sigma_\xi \sqrt{J_{t_n}(q)}}.$$

Then, $(\tilde{H}_{n,k})_{n,k\in\mathbb{N}^*}$ is a martingale difference sequence (for the obvious filtration) which satisfies the conditions of the central limit theorem for such sequences (see e.g. Theorem 8 p.442 in [Liptser and Shiryayev, 1989]). Thus,

$$\frac{H_{t_n}(q)}{\sigma_\xi \sqrt{J_{t_n}(q)}} = \sum_{k=1}^{n} \tilde{H}_{n,k} \xrightarrow{d} \xi$$

as $n \to \infty$, where $\xi \overset{d}{=} \mathcal{N}(0,1)$. Thus,

$$-\frac{1}{\sqrt{J_{t_n}(q)}} \int_{0+}^{t_n} \frac{dX_s}{q_{s-}} \xrightarrow{d} -a_X L(q) + \sigma_\xi \xi$$

as $n \to \infty$. Then, by the Skorokhod representation theorem, we can find a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$, a random variable $\tilde{\xi}$ and a process $\tilde{X}$ which are equal in law to the random variable $\xi$ and the process $X$ respectively such that

$$\lim_{n\to\infty} -\frac{1}{\sqrt{J_{t_n}(q)}} \int_{0+}^{t_n} \frac{d\tilde{X}_s}{q_{s-}} = -a_X L(q) + \sigma_\xi \tilde{\xi} \ \ (\tilde{\mathbf{P}} - a.s.).$$

Thus, on the set $\{-a_X L(q) + \sigma_\xi \tilde{\xi} > 0\}$ of positive probability, we have

$$\limsup_{t\to\infty} \left( -\int_{0+}^{t} \frac{d\tilde{X}_s}{q_{s-}} \right) \ge \lim_{n\to\infty} \left( -\int_{0+}^{t_n} \frac{d\tilde{X}_s}{q_{s-}} \right) = \infty,$$

and so also

$$\mathbf{P}\left( \limsup_{t\to\infty} \left( -\int_{0+}^{t} \frac{dX_s}{q_{s-}} \right) = \infty \right) > 0.$$

So, this last probability is equal to one for all $q \in Q$. But, since $Q$ is itself an event of probability 1, we finally obtain

$$\mathbf{P}\left( \limsup_{t\to\infty} \left( -\int_{0+}^{t} \frac{dX_s}{\mathcal{E}(R)_{s-}} \right) = \infty \right) = 1.$$

$\square$

Finally, we apply this result to the case of Lévy processes.

*Proof of Corollary 3.5.2.* Assumption (3.14) implies that $\mathbf{E}(|\hat{R}_1|) < \infty$ and, by the law of large numbers for Lévy processes, we get that

$$\lim_{t\to\infty} \frac{\hat{R}_t}{t} = \mathbf{E}(\hat{R}_1) = a_R - \frac{\sigma_R^2}{2} + \int_{-1}^{\infty} (\ln(1+x) - x\mathbf{1}_{\{|\ln(1+x)|\leq 1\}})K_R(dx).$$

But, the fact that $\lim_{t\to\infty} \frac{\hat{R}_t}{t} < 0$ is equivalent to $I_\infty = J_\infty = \infty$ ($\mathbf{P} - a.s.$) by Theorem 1 in [Bertoin and Yor, 2005]. So it is enough to check that the limit

$$\lim_{t\to\infty} \frac{I_t}{\sqrt{J_t}} = L$$

exists with $0 < L < \infty$ ($\mathbf{P} - a.s.$).

We obtain, using de l'Hospital's rule and the time-reversion property of $\hat{R}$ that

$$\lim_{t\to\infty} \frac{I_t}{\sqrt{J_t}} = \lim_{t\to\infty} 2e^{\hat{R}_t}\sqrt{J_t} \stackrel{d}{=} 2\left(\int_0^\infty e^{2\hat{R}_s} ds\right)^{1/2}$$

The last integral is finite ($\mathbf{P} - a.s.$) again by Theorem 1 in [Bertoin and Yor, 2005]. □

# Chapter 4

# On the Law at Fixed Time of GOU Processes

In this chapter, we assume that $X$ and $R$ are independent Lévy processes with triplets $(a_X, \sigma_X^2, K_X)$ and $(a_R, \sigma_R^2, K_R)$, and such that $\Delta R_t = R_t - R_{t-} > -1$, for all $t \geq 0$ ($\mathbf{P} - a.s.$). We have seen (in Section 1.3) that the GOU process $Y = (Y_t)_{t \geq 0}$ given by

$$Y_t = \mathcal{E}(R)_t \left( y + \int_{0+}^{t} \mathcal{E}(R)_{s-}^{-1} dX_s \right), \quad t \geq 0, \tag{4.1}$$

where $\mathcal{E}(.)$ is Doléans-Dade's exponential (see Proposition 1.2.10 for the definition) is the unique (strong) solution to the following SDE :

$$dY_t = dX_t + Y_{t-} dR_t, \quad t \geq 0,$$

with $Y_0 = y \geq 0$ and is used as a model for the surplus of an insurance company facing both market and insurance risks. In the last chapter, we studied the ruin probabilities of this process but we also mentioned that today the risk of an enterprise is calculated mostly via measures based on the quantiles of the distribution of $Y_t$, for some fixed time $t \geq 0$ (this is closely related to the so-called *value-at-risk*[1]). Thus, in this chapter, we give some directions for the study of the law of $Y_t$.

[1] See e.g. [McNeil et al., 2015] for the definition and related risk measures.

We start this chapter by recalling some related results in Section 4.1. Then, in Section 4.2, we use standard methods of the theory of Markov processes to obtain a partial integro-differential equation for the density when it exists. We also give sufficient conditions for the existence of this density. Since this equation seems hard to solve, we concentrate in the remaining parts on approximations for the law of $Y_t$ when $t$ is either small or large. Following this idea, we obtain, in Section 4.3, a normal-log-normal mixture approximation of the law of $Y_t$ when $t$ is small, and, in Section 4.4, log-normal approximations for the negative and positive parts of the law of $Y_t$ when $t$ is large, in the so-called small-volatility case.

## 4.1 Related Results

The question of the identification of the law of $Y_t$, for fixed $t \geq 0$, seems to be quite open. There are, however, some results in the literature which we mention now. In [Hadjiev, 1985], the Laplace transform of $Y_t$ is computed when $Y$ is the LOU process (see Section 1.3 for the definition) and a slight adaptation of the proof yields the characteristic function for these processes. However, these expressions seems hard to invert even in this simpler case. In [Feng et al., 2019], the case when $R_t = t$, for all $t \geq 0$, and when $X$ is a Lévy process is studied. The authors obtain a formula for the Mellin transform of $Y_{e(q)}$, where $e(q)$ is an independent exponential random variable with rate $q > 0$ and they are able to invert the Mellin transform to obtain the density of $Y_{e(q)}$ when, in addition, $X$ is the Kou process. Lastly, in [Brokate et al., 2008] a partial integro-differential equation is obtained for $f(t,x) = \mathbf{P}(Y_t > x)$ in the particular case when $X$ is the Cramér-Lundberg model, $R$ is a Lévy process and $y = 0$.

In contrast with the identification of the law of $Y_t$, the question of the existence and the identification of the stationary law of $Y$ which is, under some conditions (see Lemma 4.4.1 below), the random variable

$$Z_\infty = \int_{0+}^{\infty} \mathcal{E}(R)_{s-} dX_s,$$

has received a great deal of attention. This is due to the fact that $Z_\infty$ is the natural generalization of the exponential functional

$$I_\infty = \int_0^\infty \mathcal{E}(R)_{s-} ds$$

which appears in numerous domains of probability (see [Bertoin and Yor, 2005] for some examples). In [Carmona, 1996], under some conditions, partial integro-differential equations are obtained for the characteristic function and the density (when it exists) of $Z_\infty$. These results are treated more extensively and generalized in [Behme and Lindner, 2015], [Carmona et al., 2001] and [Gjessing and Paulsen, 1997]. We mention specifically [Gjessing and Paulsen, 1997] where a list of explicit distributions for $Z_\infty$ is given for different choices of $X$ and $R$. The are many other more subtle results available on $Z_\infty$ such as a characterisation of the almost sure finiteness, conditions for the infinite decomposability and conditions for the existence of the density of its law. Since they are less related to our concern in this chapter, we refer the interested reader to [Behme, 2015] and [Bertoin et al., 2008] and the references therein for a more extensive discussion.

In this chapter, we thus propose the study of $Y_t$ as an interesting problem in itself and lay some directions which expand the mentioned literature.

## 4.2 Kolmogorov-type Equation and Existence of the Density

Recall that if $S = (S_t)_{t\geq 0}$ is a time-homogeneous Markov process on $\mathbb{R}$, we define its transition semi-group $(P_t)_{t\geq 0}$ as $P_t f(x) = \mathbf{E}_x(S_t)$, for $x \in \mathbb{R}$ and $f \in \mathcal{C}(\mathbb{R})$ and where $\mathbf{E}_x(.)$ represents the expectation w.r.t. the law of $S_t$ conditionally to $S_0 = x$. For the Markov process $S$ we can then define the *domain of the generator* $D^S$ as the set of $f \in \mathcal{C}(\mathbb{R})$ such that there exists a limit

$$\mathcal{A}^S f = \lim_{t\to 0} \frac{P_t f - f}{t}$$

w.r.t. the uniform topology on $\mathcal{C}(\mathbb{R})$ (i.e. the topology given by the uniform norm $|g|_\infty = \sup_{x\in\mathbb{R}} |g(x)|$).[2] For $f \in D^S$, the operator $\mathcal{A}^S$ is called the *generator* of $S$.

[2] We refer to Section 1.1 for the definition of the functional spaces used in this chapter.

*Feller processes* are time-homogeneous Markov process for which the generator satisfies some additional regularity conditions. Following the convention of [Behme and Lindner, 2015], we define Feller processes as follows.

**Definition 4.2.1.** *A time-homogeneous Markov process $(S_t)_{t\geq 0}$ is a Feller process if its transition semi-group $(P_t)_{t\geq 0}$ satisfies, for each $f \in \mathcal{C}_0(\mathbb{R})$,*

1. $P_t f \in \mathcal{C}_0(\mathbb{R})$,
2. $\lim_{t\to 0} P_t f = f$, *for the uniform topology on* $\mathcal{C}(\mathbb{R})$.

In [Behme and Lindner, 2015] it was shown that the GOU process (4.1) is a Feller process and an expression for the generator is given.

**Proposition 4.2.2** (Corollary 3.2 in [Behme and Lindner, 2015])**.** *The process $Y$ is a Feller process with generator $\mathcal{A}^Y$ whose domain contains the set*

$$S = \{f \in \mathcal{C}_0^2(\mathbb{R}) : \lim_{x\to\infty} (|xf'(x)| + |x^2 f''(x)|) = 0\},$$

*and which, for $f \in S$, is given by*

$$\begin{aligned} \mathcal{A}^Y f(x) = &\frac{1}{2}(\sigma_X^2 + \sigma_R^2 x^2) f''(x) + (a_X + a_R x) f'(x) \\ &+ \int_{-1}^{\infty} (f(x+xz) - f(x) - f'(x)xz\mathbf{1}_{\{|z|\leq 1\}}) K_R(dz) \\ &+ \int_{\mathbb{R}} (f(x+z) - f(x) - f'(x)z\mathbf{1}_{\{|z|\leq 1\}}) K_X(dz), \end{aligned} \tag{4.2}$$

*where $K_X$ and $K_R$ are the Lévy measures of the processes $X$ and $R$.*

From this expression, we can now obtain an equation for the density (when it exists) using the standard theory of Feller processes and integration by parts. This is a similar method to the one used to obtain the equation for the density of exponential functionals of processes with independent increments in [Vostrikova, 2018]. We denote by $p(t,x)$ the density of the law of $Y_t$ w.r.t. the Lebesgue measure and denote $p = (p(t,x))_{t\geq 0, x\in\mathbb{R}}$.

**Theorem 4.2.3.** *Assume that $K_R((-1,\infty)) < \infty$ and that $K_X(\mathbb{R}) < \infty$. Assume that $Y_t$ admits a density $p \in \mathcal{C}^{1,2}((0,T)\times\mathbb{R})$ w.r.t. the Lebesgue* *measure $\lambda$, for all $t \in (0,T)$. Then, $p$ satisfies the following partial integro-differential equation :*

$$\begin{aligned}
&\partial_t p(t,x) = \\
&\quad \frac{\sigma_X^2}{2}\partial_{xx}p(t,x) + \frac{\sigma_R^2}{2}\partial_{xx}\left(x^2 p(t,x)\right) - a_X\partial_x p(t,x) - a_R\partial_x(xp(t,x)) \\
&\quad + \int_{-1}^{\infty}\left(\frac{p(t,x(1+z)^{-1})}{1+z} - p(t,x) + z\mathbf{1}_{\{|z|\leq 1\}}\partial_x(xp(t,x))\right)K_R(dz) \\
&\quad + \int_{\mathbb{R}}\left(p(t,x-z) - p(t,x) + z\mathbf{1}_{\{|z|\leq 1\}}\partial_x p(t,x)\right)K_X(dz)
\end{aligned}$$

*for all $(t,x) \in (0,T)\times\mathbb{R}$, with initial condition $p(0,x) = \delta_y(x)$, where $\delta_y$ is the Dirac measure at $y > 0$.*

*Proof.* Condition 2 of Definition 4.2.1 means that the transition semi-group is strongly continuous on $\mathcal{C}_0(\mathbb{R})$. Moreover, by Proposition 1.5 p.9 in [Ethier and Kurtz, 1986], we then know that for $f \in D^Y$, we have $P_t f \in D^Y$, for each $t \geq 0$, and

$$\frac{d}{dt}P_t f = \mathcal{A}^Y P_t f = P_t\mathcal{A}^Y f. \tag{4.3}$$

Let $M_t = f(Y_t) - \int_0^t \mathcal{A}^Y f(Y_u)du$, for all $t \geq 0$ and $f \in D^Y$. We will prove first that $(M_t)_{t\geq 0}$ is a martingale w.r.t. the natural filtration $(\mathcal{F}_t^Y)_{t\geq 0}$ of $Y$. The integrability follows from the fact that $f$ and $\mathcal{A}^Y f$ are continuous functions vanishing at infinity and thus are bounded. For the martingale property, note that, for $0 \leq s < t$, we obtain $(\mathbf{P} - a.s.)$ using the change of variables $v = u - s$ and Fubini's theorem,

$$\begin{aligned}
\mathbf{E}(M_t - M_s|\mathcal{F}_s^Y) &= \mathbf{E}(f(Y_t) - f(Y_s)|\mathcal{F}_s^Y) - \int_s^t \mathbf{E}(\mathcal{A}^Y f(Y_u)|\mathcal{F}_s^Y)du \\
&= \mathbf{E}(f(Y_t) - f(Y_s)|\mathcal{F}_s^Y) - \int_0^{t-s} \mathbf{E}(\mathcal{A}^Y f(Y_{v+s})|\mathcal{F}_s^Y)dv.
\end{aligned}$$

But, from the Markov property and Equation (4.3), we obtain $(\mathbf{P} - a.s.)$

$$\mathbf{E}(\mathcal{A}^Y f(Y_{v+s})|\mathcal{F}_s^Y) = \mathbf{E}(\mathcal{A}^Y f(Y_{v+s})|Y_s) = P_v\mathcal{A}^Y f(Y_s) = \frac{d}{dv}P_v f(Y_s).$$

Thus, ($\mathbf{P}-a.s.$)

$$\begin{aligned}\mathbf{E}(M_t - M_s|\mathcal{F}_s^Y) &= \mathbf{E}(f(Y_t)|Y_s) - f(Y_s) - \int_0^{t-s} \frac{d}{dv}P_v f(Y_s)dv \\ &= P_{t-s}f(Y_s) - f(Y_s) - \int_0^{t-s} \frac{d}{dv}P_v f(Y_s)dv = 0.\end{aligned}$$

This means that $(M_t)_{t\geq 0}$ is a martingale and $\mathbf{E}(M_t) = \mathbf{E}(M_0) = 0$, for all $t \geq 0$, and thus, if $Y_t$ admits a density $p(t,.)$, that

$$\frac{d}{dt}\mathbf{E}[f(Y_t)] = \int_{\mathbb{R}} f(x)\partial_t p(t,x)dx = \int_{\mathbb{R}} \mathcal{A}^Y f(x)p(t,x)dx, \tag{4.4}$$

for all $f \in D^Y$.

Let $\mathcal{C}_c^2$ designate the functions of class $\mathcal{C}^2$ with compact support. Then, $\mathcal{C}_c^2 \subset \mathcal{S}$ and, using the definition of the generator of $Y$, we find that for $f \in \mathcal{C}_c^2$,

$$\int_{\mathbb{R}} \mathcal{A}^Y f(x)p(t,x)dx = I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 = \int_{\mathbb{R}} \frac{1}{2}(\sigma_X^2 + \sigma_R^2 x^2)f''(x)p(t,x)dx,$$

$$I_2 = \int_{\mathbb{R}} (a_X + a_R x)f'(x)p(t,x)dx,$$

$$I_3 = \int_{\mathbb{R}} \int_{-1}^{\infty} p(t,x)(f(x+xz) - f(x) - f'(x)xz\mathbf{1}_{\{|z|\leq 1\}})K_R(dz)dx$$

and

$$I_4 = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} (f(x+z) - f(x) - f'(x)z\mathbf{1}_{\{|z|\leq 1\}})p(t,x)dx \right) K_X(dz).$$

Using integration by parts, we obtain

$$I_1 = \int_{\mathbb{R}} \left( \frac{\sigma_X^2}{2}\partial_{xx}p(t,x) + \frac{\sigma_R^2}{2}\partial_{xx}\left(x^2 p(t,x)\right) \right) f(x)dx.$$

and

$$I_2 = -\int_{\mathbb{R}} \left(a_X \partial_x p(t,x) + a_R \partial_x \left(xp(t,x)\right)\right) f(x)dx.$$

Using the change of variables $u = (1+z)x$ and $K_R((-1,\infty)) < \infty$, we obtain

$$\begin{aligned} I_3 = &\int_{\mathbb{R}} \left( \int_{-1}^{\infty} p(t, u(1+z)^{-1})(1+z)^{-1} K_R(dz) \right) f(u) du \\ &- \int_{\mathbb{R}} \left( \int_{-1}^{\infty} p(t,x) K_R(dz) \right) f(x) dx \\ &+ \int_{\mathbb{R}} \left( \int_{-1}^{\infty} z \mathbf{1}_{\{|z| \le 1\}} \partial_x (x p(t,x)) K_R(dz) \right) f(x) dx. \end{aligned}$$

Using the change of variables $u = x + z$ and $K_X(\mathbb{R}) < \infty$, we obtain

$$\begin{aligned} I_4 = &\int_{\mathbb{R}} \left( \int_{\mathbb{R}} p(t, u-z) K_X(dz) \right) f(u) du - \int_{\mathbb{R}} \left( \int_{\mathbb{R}} p(t,x) K_X(dz) \right) f(x) dx \\ &+ \int_{\mathbb{R}} \left( \int_{\mathbb{R}} z \mathbf{1}_{\{|z| \le 1\}} \partial_x p(t,x) K_X(dz) \right) f(x) dx. \end{aligned}$$

Since Equation (4.4) is valid for all $f \in \mathcal{C}_c^2$, this yields the claimed equation ($\lambda - a.s.$). □

The theorem gives no information about the existence of the density of $Y_t$, however. Thus, we give, in the next proposition, some simple conditions for the existence of a sufficiently regular density and thus for the existence of a classical solution to the PIDE from Theorem 4.2.3. We recall that $Y_t$ depends on $y$ its initial value and so does $p$. To make this dependence explicit, we will write $p^y(t,x)$ instead of $p(t,x)$ from now on.

**Proposition 4.2.4.** *Assume that*

1. $\mathbf{E}(|X_1|^p) < \infty$ *and* $\mathbf{E}(|R_1|^p) < \infty$, *for all* $p \in \mathbb{N}^*$,
2. $\sigma_X > 0$,
3. *there exists* $\rho > 0$ *such that* $K_R((-\infty, -1+\rho]) = 0$.

*Then, for all* $T > 0$, *the function* $(y,t,x) \mapsto p^y(t,x)$ *is of class* $\mathcal{C}^\infty(\mathbb{R}_+^* \times (0,T) \times \mathbb{R})$.

*Proof.* Recall that the Lévy-Itô decompositions (1.11) of $X$ and $R$ are:

$$X_t = \delta_X t + \sigma_X W_t^{(1)} + \int_{0+}^t x(\mu_X(ds,dx) - K_X(dx)ds) \tag{4.5}$$

and

$$R_t = \delta_R t + \sigma_R W_t^{(2)} + \int_{0+}^{t} x(\mu_R(ds,dx) - K_R(dx)ds),$$

where $\delta_X = \mathbf{E}(X_1)$ and $\delta_R = \mathbf{E}(R_1)$ and $(W_t^{(1)})_{t\geq 0}$ and $(W_t^{(2)})_{t\geq 0}$ are two independent Brownian motions.

Since the processes $X$ and $R$ are independent, the process $L = (L_t)_{t\geq 0}$ given by

$$L_t = \begin{pmatrix} \int_{0+}^{t} \int_{\mathbb{R}} x(\mu_X(ds,dx) - K_X(dx)ds) \\ \int_{0+}^{t} \int_{-\infty}^{1} x(\mu_R(ds,dx) - K_R(dx)ds) \end{pmatrix},$$

for all $t \geq 0$, is a two dimensional Lévy process. Thus, its jump measure $\mu_L$ is a Poisson random measure on $E = \mathbb{R} \times (-\infty, 1)$ and its Lévy measure is $K_L(dz_1, dz_2) = K_X(dz_1)\delta_0(z_2)dz_2 + K_R(dz_2)\delta_0(z_1)dz_1$, where $\delta_0$ is the Dirac measure at 0. The expression of $K_L$ follows from the fact that since $X$ and $R$ are independent they do not jump together and, thus, the jumps of $L$ are caused either by the jumps of $X$ or $R$, i.e., the number of jumps of $L$ of sizes belonging to $A \subset E$ is the sum of the number of jumps of $X$ of sizes belonging to $A \cap \mathbb{R} \times \{0\}$ and the number of jumps of $R$ of sizes belonging to $A \cap \{0\} \times (-\infty, 1)$.

Thus, the process $Y$ can be rewritten as the solution to the following SDE:

$$\begin{aligned} Y_t = y &+ \int_0^t a(Y_{s-})ds + \int_{0+}^t b(Y_{s-})dW_s \\ &+ \int_{0+}^t \int_E c(Y_{s-}, z_1, z_2)(\mu_L(ds, dz_1, dz_2) - K_L(dz_1, dz_2)ds), \end{aligned} \tag{4.6}$$

where $W_t = (W_t^{(1)}, W_t^{(2)})^T$ is a two-dimensional Brownian motion, $a(u) = \delta_X + \delta_R u$, $b(u) = (\sigma_X, \sigma_R u) \in \mathcal{M}_{1,2}(\mathbb{R})$ and $c(u, z_1, z_2) = z_1 + uz_2$ are appropriately chosen functions. Here $\mathcal{M}_{1,2}(\mathbb{R})$ is the space of real-valued matrices with 1 row and 2 columns and the product $b(Y_{s-})dW_s$ has to be interpreted as the scalar product between $b(Y_{s-})$ and $(dW_t^{(1)}, dW_t^{(2)})^T$.

To prove the existence of the density, we will now use Theorem 2-29 in [Bichteler et al., 1987], which gives conditions for the existence of densities for solutions of SDEs of the form (4.6). In order to state the theorem we introduce the following conditions which are adapted to our problem.

The first condition is $(\mathbf{A} - \mathbf{r})$:

- $a$, $b$ are $r$-times differentiable with bounded derivatives of all orders,
- $c$ is $\mathbb{R}_+ \times E$-measurable, $c(., z_1, z_2)$ is $r$-times differentiable for each $(z_1, z_2) \in E$, $c(0, .) \in \cap_{p=2}^{\infty} \mathcal{L}^p(E, K_L)$ and
$$\sup_{u \in \mathbb{R}} \left| \left( \frac{\partial}{\partial u} \right)^n c(u, .) \right| \in \bigcap_{p=2}^{\infty} \mathcal{L}^p(E, K_L).$$
for all $1 \le n \le r$, where, for an integer $p \ge 2$, the notation $\mathcal{L}^p(E, K_L)$ denotes the class of functions on $E$ whose $p$-th power is integrable w.r.t. to $K_L$.

The second condition is $(\mathbf{B} - (\epsilon, \delta))$: there exists $\epsilon > 0$ and $\delta \ge 0$ such that, for all $u \in \mathbb{R}$, we have
$$b(u)b(u)^T \ge \frac{\epsilon}{1 + |u|^\delta}.$$

And the last condition is $(\mathbf{C} - \boldsymbol{\Xi})$: there exists $\Xi > 0$ such that
$$\left| 1 + w \frac{\partial}{\partial u} c(u, z_1, z_2) \right| \ge \Xi,$$
for all $w \in [0, 1]$, $u \in \mathbb{R}$ and $(z_1, z_2) \in E$.

Fixing $r \in \mathbb{N}^*$ and $T > 0$, Theorem 2-29 p.15 in [Bichteler et al., 1987], shows that if the conditions $(\mathbf{A} - (\mathbf{2r} + \mathbf{10}))$, $(\mathbf{B} - (\epsilon, \delta))$ and $(\mathbf{C} - \boldsymbol{\Xi})$ are satisfied, then the function $(y, t, x) \mapsto p^y(t, x)$ is of class $\mathcal{C}^r(\mathbb{R}_+^* \times (0, T) \times \mathbb{R})$.

Concerning condition $(\mathbf{A} - (\mathbf{2r} + \mathbf{10}))$ we see that, in our case, it is equivalent to
$$\int_E |z_1|^p K_L(dz_1, dz_2) < \infty \text{ and } \int_E |z_2|^p K_L(dz_1, dz_2) < \infty,$$
which in turn is equivalent to
$$\int_{\mathbb{R}} |z_1|^p K_X(dz_1) < \infty \text{ and } \int_{-\infty}^{1} |z_2|^p K_R(dz_2) < \infty,$$
or $\mathbf{E}(|X_1|^p) < \infty$ and $\mathbf{E}(|R_1|^p) < \infty$, for all integers $p \ge 2$, by Theorem 25.3 p.159 in [Sato, 1999]. This shows that $(\mathbf{A} - (\mathbf{2r} + \mathbf{10}))$ is satisfied, for all $r \in \mathbb{N}^*$.

Condition $(\mathbf{B}-(\epsilon,\delta))$ is satisfied with $\delta=0$ and $\epsilon=\sigma_X^2$, since $b(u)b(u)^T=\sigma_X^2+\sigma_R^2u^2\geq\sigma_X^2$, for all $u\in\mathbb{R}$.

Finally, for the last condition, note that, since $z_2\geq -1+\rho$ by assumption 3, we have

$$1+w\frac{\partial}{\partial u}c(u,z_1,z_2)=1+wz_2\geq 1+(\rho-1)w.$$

Now if $\rho\geq 1$, we take $\Xi=1$ and the proof is finished. If $\rho<1$, the function $w\mapsto 1+(\rho-1)w$ is non-increasing and $1+w\frac{\partial}{\partial u}c(u,z_1,z_2)\geq\rho$. Thus, we can take $\Xi=\rho$ to satisfy condition $(\mathbf{C}-\boldsymbol{\Xi})$ and to finish the proof. □

The assumptions in the last proposition are somewhat restrictive, but they allow to formulate the following proposition which describes the density of the pure diffusion model.

**Proposition 4.2.5** (Fokker-Plank Equation in the Pure Diffusion Case)**.** *Assume that $X_t=a_Xt+\sigma_XW_t^{(1)}$ and $R_t=a_Rt+\sigma_RW_t^{(2)}$, for all $t\geq 0$, where $W^{(1)}=(W_t^{(1)})_{t\geq 0}$ ans $W^{(2)}=(W_t^{(2)})_{t\geq 0}$ are two independent Brownian motions. Assume that $\sigma_X>0$. Then, $Y_t$ (with $Y_0=y>0$) admits a density $p^y$ such that $(y,t,x)\mapsto p^y(t,x)$ is of class $\mathcal{C}^\infty(\mathbb{R}_+^*\times(0,T)\times\mathbb{R})$, for any $T>0$, and it is the unique solution of the following PDE :*

$$\begin{aligned}\partial_tp^y(t,x)=\frac{\sigma_X^2}{2}\partial_{xx}p^y(t,x)+\frac{\sigma_R^2}{2}\partial_{xx}\left(x^2p^y(t,x)\right)-a_X\partial_xp^y(t,x)\\ -a_R\partial_x\left(xp^y(t,x)\right)\end{aligned}$$

*with $(t,x)\in(0,T)\times\mathbb{R}$ and initial condition $p^y(0,x)=\delta_y(x)$.*

*Proof.* The existence of the density follows from Proposition 4.2.4 and the form of the equation from Theorem 4.2.3. The only thing that remains to be checked is the uniqueness of the solution.

Note that we can rewrite the equation in the parabolic form

$$\partial_tp^y(t,x)-a(x)\partial_{xx}p^y(t,x)+b(x)\partial_xp^y(t,x)+c(x)p^y(t,x)=0$$

with

$$a(x)=\frac{1}{2}(\sigma_X^2+\sigma_R^2x^2),$$
$$b(x)=2\sigma_R^2-a_X-a_Rx$$

and

$$c(x) = \sigma_R^2 - a_R.$$

We also have

$$a(x) = \frac{1}{2}(\sigma_X^2 + \sigma_R^2 x^2) \geq \frac{\sigma_X^2}{2} > 0,$$

for all $x \in \mathbb{R}$, and, thus, the equation is uniformly parabolic in the sense of the definition p.350 in [Evans, 1998]. Thus, the uniqueness follows from the general uniqueness result for this type of equation, e.g. Theorem 4 p.358 in [Evans, 1998]. □

We thus obtained an equation that yields as a solution the density of $Y_t$, for each fixed time $t \geq 0$, and showed that, in the pure diffusion case at least, the solution to the equation in Theorem 4.2.3 is exactly this density. However, these results seem to be of little practical value since, even in the pure diffusion case, the equations seem hard to integrate without some numerical method. We will thus focus in the remaining parts on approximations for the laws of $Y_t$, when $t$ is either small or large.

## 4.3 Small-time Approximation

In this section, we will suggest an approximating distribution for $Y_t$ when $t$ is small. This could be useful in the banking business where risk measures such as value-at-risk are usually computed overnight.

Let $\hat{R} = (\hat{R}_t)_{t\geq 0}$ be the exponential transform of $R$. Then, by definition, we have

$$Y_t = e^{\hat{R}_t}\left(y + \int_{0+}^t e^{-\hat{R}_{s-}} dX_s\right), \ \ t \geq 0.$$

Define also the exponential functional of $\hat{R}$ by

$$I_t = \int_0^t e^{\hat{R}_{s-}} ds, \ \ t \geq 0.$$

The approximations we obtain will depend themselves on an approximation of the exponential functional. In order to have an explicit approximation for this functional, we restrict ourselves to the case where $R_t = a_R t + \sigma_R W_t^{(2)}$

and, hence, $\hat{R}_t = d_R t + \sigma_R W_t^{(2)}$ with $d_R = a_R - \sigma_R^2/2$.

**Theorem 4.3.1.** *Assume that $R_t = a_R t + \sigma_R W_t^{(2)}$, for all $t \geq 0$, with $\sigma_R > 0$, and that*

$$\int_{\mathbb{R}} |x| K_X(dx) < \infty. \tag{4.7}$$

*Then, as $t \to 0+$,*

$$\frac{Y_t - y - \delta_X I_t}{\sqrt{\sigma(y) I_t}} \xrightarrow{d} \mathcal{N}(0,1),$$

*where $\sigma(y) = \sigma_X^2 + \sigma_R^2 y^2$ and $\delta_X = \mathbf{E}(X_1)$.*

*Proof.* First note that, for each $t > 0$,

$$(I_t, Y_t) \stackrel{d}{=} \left( I_t, y e^{\hat{R}_t} + \int_{0+}^t e^{\hat{R}_{s-}} dX_s \right). \tag{4.8}$$

This follows from the time-reversal property of $X$ and $\hat{R}$, namely, for each $t \geq 0$, we have

$$(\hat{R}_t, (\hat{R}_t - \hat{R}_{(t-s)-})_{0 \leq s \leq t}, (X_t - X_{(t-s)-})_{0 \leq s \leq t}) \stackrel{d}{=} (\hat{R}_t, (\hat{R}_s)_{0 \leq s \leq t}, (X_s)_{0 \leq s \leq t})$$

and so

$$Y_t = y e^{\hat{R}_t} + \int_{0+}^t e^{\hat{R}_t - \hat{R}_{s-}} dX_s \stackrel{d}{=} y e^{\hat{R}_t} + \int_{0+}^t e^{\hat{R}_{s-}} dX_s.$$

The condition (4.7) implies that $X$ is integrable, and so, using the Lévy-Itô decomposition (4.5) and Proposition 3.2.4, we obtain

$$\left( \int_{0+}^t e^{\hat{R}_{s-}} dX_s \right)_{t \geq 0} \stackrel{d}{=} \left( \delta_X I_t + \sigma_X \int_0^t e^{\hat{R}_{s-}} dW_s^{(1)} + N_t^d \right)_{t \geq 0} \tag{4.9}$$

with

$$N_t^d = \int_{0+}^t \int_{\mathbb{R}} e^{\hat{R}_{s-}} x (\mu_X(ds, dx) - K_X(dx) ds),$$

and where $W^{(1)}$ and $N^d$ are independent.

Let

$$V_t = \frac{Y_t - y - \delta_X I_t}{\sqrt{\sigma(y) I_t}}, \quad t \geq 0.$$

We will compute the characteristic function of $V_t$ and show that it tends to the characteristic function of a standard normal distribution as $t \to 0+$.

We have, for all $u \in \mathbb{R}$,

$$\begin{aligned}&\mathbf{E}\left[\exp\left(iuV_t\right)\right] = \\ &\quad \mathbf{E}\left[\mathbf{E}\left(\exp\left(iu\frac{Y_t - y - \delta_X I_t}{\sqrt{\sigma(y)I_t}}\right)\middle| \hat{R}_s = g(s), s \geq 0\right)\right].\end{aligned} \tag{4.10}$$

For any $g \in \mathbb{D}$, denote $I_t(g) = \int_0^t e^{g(s-)}ds,\ J_t(g) = \int_0^t e^{2g(s-)}ds$ and

$$N_t^d(g) = \int_{0+}^t \int_{\mathbb{R}} e^{g(s-)}x(\mu_X(ds,dx) - K_X(dx)ds).$$

Using the independence between $X$ and $R$, the conditional independence between $W^{(1)}$ and $N^d$ and Equations (4.8) and (4.9), we find that the conditional expectation in (4.10) is equal to

$$\begin{aligned}&\mathbf{E}\left(\exp\left(iu\frac{y(e^{g(t)}-1) + \sigma_X \int_{0+}^t e^{g(s-)}d\tilde{W}_s + N_t^d(g)}{\sqrt{\sigma(y)I_t(g)}}\right)\right) \\ &\quad = \exp\left(iu\frac{y(e^{g(t)}-1)}{\sqrt{\sigma(y)I_t(g)}}\right)\mathbf{E}\left(\exp\left(iu\frac{\sigma_X\int_{0+}^t e^{g(s-)}dW_s^{(1)}}{\sqrt{\sigma(y)I_t(g)}}\right)\right) \\ &\quad \times \mathbf{E}\left(\exp\left(iu\frac{N_t^d(g)}{\sqrt{\sigma(y)I_t(g)}}\right)\right).\end{aligned}$$

Since $\int_{0+}^t e^{g(s-)}dW_s^{(1)}$ is a normal random variable with variance $J_t(g)$, the first expectation in the r.h.s. of the above equation is equal to

$$\exp\left(-\frac{u^2\sigma_X^2}{2\sigma(y)}\frac{J_t(g)}{I_t(g)}\right).$$

For the second one we obtain, using the compensation theorem (which is allowed by (4.7)) and the expression for the characteristic function of integrals w.r.t. Poisson random measures see e.g. Proposition 19.5 p.123 in [Sato, 1999],

$$\begin{aligned}
&\mathbf{E}\left(\exp\left(iu[\sigma(y)I_t(g)]^{-1/2}N_t^d(g)\right)\right)\\
&\quad=\exp\left(\int_{0+}^t\int_{\mathbb{R}}\left(e^{iu[\sigma(y)I_t(g)]^{-1/2}e^{g(s-)}x}-\frac{iue^{g(s-)}x}{\sqrt{\sigma(y)I_t(g)}}-1\right)K_X(dx)ds\right)\\
&\quad=\exp\left(\int_{0+}^t\int_{\mathbb{R}}H(x,\hat{R}_{s-},I_t(g))K_X(dx)ds\right),
\end{aligned}$$

where

$$H(x,\hat{R}_{s-},I)=\exp(iu[\sigma(y)I]^{-1/2}e^{\hat{R}_{s-}}x)-\frac{iue^{\hat{R}_{s-}}x}{\sqrt{\sigma(y)I}}-1.$$

Combining the above computations, we get

$$\begin{aligned}
&\mathbf{E}\left[\exp\left(iuV_t\right)\right]=\\
&\quad\mathbf{E}\left(\exp\left(iu\frac{y(e^{\hat{R}_t}-1)}{\sqrt{\sigma(y)I_t}}-\frac{u^2\sigma_X^2}{2\sigma(y)}\frac{J_t}{I_t}+\int_{0+}^t\int_{\mathbb{R}}H(x,\hat{R}_{s-},I_t)K_X(dx)ds\right)\right).
\end{aligned}$$

Let

$$h(t)=\int_{0+}^t\int_{\mathbb{R}}H(x,\hat{R}_{s-},I_t)K_X(dx)ds,\ \ t\geq 0.$$

By an easy computation, we see that, for all $\alpha\in\mathbb{R}$,

$$0\leq\|e^{i\alpha}-i\alpha-1\|\leq 2|\alpha|.$$

(here $\|z\|$ represents the norm of some complex number $z\in\mathbb{C}$) and so

$$\|h(t)\|\leq\int_{0+}^t\int_{\mathbb{R}}\|H(x,\hat{R}_{s-},I_t)\|K_X(dx)ds\leq\frac{2u}{\sqrt{\sigma(y)}}\left(\int_{\mathbb{R}}|x|K_X(dx)\right)\sqrt{I_t}.$$

But, $\lim_{t\to 0+}\sqrt{I_t}=0$ ($\mathbf{P}-a.s.$) and so, by (4.7), $h(t)\to 0$ ($\mathbf{P}-a.s.$), as $t\to 0+$.

Then, an application of de l'Hospital's rule yields

$$\lim_{t\to 0+}\frac{J_t}{I_t}=1\text{ and }\lim_{t\to 0+}\frac{t}{I_t}=1\ \ (\mathbf{P}-a.s.).$$

Moreover, from Taylor's approximation and since $\hat{R}_t \to 0$ ($\mathbf{P} - a.s.$), as $t \to 0+$, we have

$$\frac{e^{\hat{R}_t} - 1}{\sqrt{t}} = \frac{\hat{R}_t}{\sqrt{t}} \left(1 + o(\hat{R}_t)\right).$$

So we obtain, from Slutsky's lemma and the fact that $W_t^{(2)}/\sqrt{t} \stackrel{d}{=} \mathcal{N}(0,1)$, for all $t \geq 0$, that

$$\frac{e^{\hat{R}_t} - 1}{\sqrt{t}} \xrightarrow{d} \sigma_R \mathcal{N}(0,1),$$

as $t \to 0+$. Another application of Slutsky's lemma yields

$$\left(h(t), \frac{J_t}{I_t}, \frac{t}{I_t}, \frac{e^{R_t} - 1}{\sqrt{t}}\right) \xrightarrow{d} (0, 1, 1, \sigma_R Z),$$

as $t \to 0+$, with $Z \stackrel{d}{=} \mathcal{N}(0,1)$, and thus

$$\begin{aligned} \lim_{t \to 0+} \mathbf{E}\left[\exp\left(iuV_t\right)\right] &= \exp\left(-\frac{u^2\sigma_X^2}{2\sigma(y)}\right) \mathbf{E}\left[\exp\left(iu\frac{y\sigma_R}{\sqrt{\sigma(y)}} Z\right)\right] \\ &= \exp\left(-\frac{u^2}{2}\right). \end{aligned}$$

□

The theorem shows conditionally on the knowledge of $I_t$, we can obtain an approximation for $Y_t$ for small $t$. At least in the case when $\hat{R}_t = d_R t + \sigma_R W_t^{(2)}$, for all $t \geq 0$, an approximation of $I_t$ for small $t$ is known. Indeed, rescaling the variance of the Brownian motion from 1 to $\sigma_R^2$ in the proof of Theorem 2.2. in [Dufresne, 2004], we obtain, as $t \to 0+$,

$$\frac{\ln(I_t) - \ln(t)}{\sigma_R \sqrt{t/3}} \xrightarrow{d} \mathcal{N}(0,1).$$

Thus, we can obtain an approximating distribution for $Y_t$ when $t$ is small, as a normal-log-normal mean-variance mixture with common mixing random variable. We refer to [Barndorff-Nielsen et al., 1982] for more details about such distributions.

**Small-time Approximating Distribution.** Under the assumptions of Theorem 4.3.1, we suggest to use a normal-log-normal mixture model defined by the following density

$$f_{NLN}(u,t) = \int_0^\infty \frac{1}{\sqrt{2\pi\sigma(y)v}} \exp\left(-\frac{(u-y-\delta_X v)^2}{2\sigma(y)v}\right) g_{LN}(v,t)dv, \quad (4.11)$$

where $\sigma(y) = \sigma_X^2 + \sigma_R^2 y^2$ and

$$g_{LN}(v,t) = \frac{1}{\sigma_R v}\sqrt{\frac{3}{2\pi t}} \exp\left(-\frac{3(\ln(v)-\ln(t))^2}{2\sigma_R^2 t}\right)$$

to approximate the law of $Y_t$ when $t$ is small.

As an example, we consider the Cramér-Lundberg model with investment in a Black-Scholes market.

**Example 4.3.2** (Cramér-Lundberg-Black-Scholes)**.** Let $X = (X_t)_{t\geq 0}$ be defined as

$$X_t = p_X t - \sum_{i=1}^{N_t} Z_i, \ \ t \geq 0,$$

where $N = (N_t)_{t\geq 0}$ is a compound Poisson process with rate $\lambda_X$ and $(Z_i)_{i\in\mathbb{N}^*}$ is an independent sequence of non-negative i.i.d. random variables with $\mathbf{E}(|Z_1|) < \infty$. The investment process is given by $R_t = a_R t + \sigma_R W_t^{(2)}$, for all $t \geq 0$, with $\sigma_R > 0$, as required in the assumptions of Theorem 4.3.1.

Let us prove that Assumption (4.7) is satisfied. It is well known that $K_X(dx) = \lambda_X F_Z(dx)$, where $F_Z$ is the distribution function of $-Z_1$ (see e.g. Proposition 3.5 p.75 in [Cont and Tankov, 2004]). Thus, we have

$$\int_{\mathbb{R}} |x| K_X(dx) = \lambda_X \mathbf{E}(|Z_1|) < \infty.$$

Thus, the assumptions of Theorem 4.3.1 are satisfied. Moreover, by conditioning on $N$ it is easy to check that $\delta_X = \mathbf{E}(X_1) = p_X - \lambda_X \mathbf{E}(Z_1)$ and we have $\sigma(y) = \sigma_R^2 y^2$, which yield the parameters of the approximating distribution.

## 4.4 Large-time Approximation

In this section, we briefly recall some known results on the large-time behaviour of $Y_t$ and obtain an log-normal approximation for the negative $(Y_t)^-$ and positive $(Y_t)^+$ parts of $Y_t$ when $t$ is large, in the small-volatility case.

Define the following quantities

$$Z_t = \int_{0+}^{t} e^{\hat{R}_{s-}} dX_s \text{ and } \tilde{Z}_t = \int_{0+}^{t} e^{-\hat{R}_{s-}} dX_s.$$

Define also $\Pi_X(x) = K_X((-\infty, -x]) + K_X([x, \infty))$. The following lemma follows readily form the known literature.

**Lemma 4.4.1.** *Assume that*

$$\mathbf{E}(|\hat{R}_1|) < \infty \textit{ and } \int_{1}^{\infty} \ln(x)|\Pi_X(dx)| < \infty.$$

1. *(Large volatility case) If $\mathbf{E}(\hat{R}_1) < 0$, then $(Z_t)_{t\geq 0}$ converges to finite random variables $Z_\infty$ $(\mathbf{P} - a.s.)$ and $Y_t \xrightarrow{d} Z_\infty$, as $t \to \infty$.*
2. *(Small volatility case) If $\mathbf{E}(\hat{R}_1) > 0$, then $(\tilde{Z}_t)_{t\geq 0}$ converges to a finite random variables $\tilde{Z}_\infty$ $(\mathbf{P} - a.s.)$.*

*Proof.* The first case is proven by applying Theorem 2 in [Erickson and Maller, 2005] to the Lévy process $(-\hat{R}_t)_{t\geq 0}$ and using Theorem 3.1 in [Carmona et al., 2001]. For the second case, we apply Theorem 2 in [Erickson and Maller, 2005] to the Lévy process $(\hat{R}_t)_{t\geq 0}$. □

**Remark 4.4.2.** The terminology of large and small volatility comes from the fact that if $R_t = a_R t + \sigma_R W_t^{(2)}$, we have $\hat{R}_t = (a_R - \sigma_R^2/2)t + \sigma_R W_t^{(2)}$, $t \geq 0$, by Proposition 1.2.12 and so $\mathbf{E}(\hat{R}_1) < 0$ is equivalent to $a_R < \sigma_R^2/2$ and $\mathbf{E}(\hat{R}_1) > 0$ to $a_R > \sigma_R^2/2$.

The lemma shows that, in the large volatility case, the known results about the distribution of $Z_\infty$ immediately give a way to approximate the distribution of $Y_t$ when $t$ is large. We refer to [Gjessing and Paulsen, 1997] where a list of the laws of $Z_\infty$ is given for various choices of $X$ and $\hat{R}$. We also mention [Paulsen and Hove, 1999] where a Markov chain Monte Carlo method is proposed to simulate $Z_\infty$.

However, the question of how to approximate $Y_t$ for large $t$ in the small volatility case seems to be open and thus the goal of this section is to obtain an approximation for that case. Part of our proof will be based on the following specialization of a result in [Feigin, 1985] which gives a central limit theorem for martingales.

**Lemma 4.4.3** (Theorem 2 in [Feigin, 1985])**.** *Let $M = (M_t)_{t\geq 0}$ be a locally square-integrable martingale. Assume that*

1. $\lim_{t\to\infty} \mathbf{E}(M_t^2) = \infty$,
2. $$\lim_{t\to\infty} \frac{\mathbf{E}(\sup_{0\leq s\leq t} |\Delta M_s|)}{\sqrt{\mathbf{E}(M_t^2)}} = 0,$$
3. *and, as $t \to \infty$,*
$$\frac{[M]_t}{\mathbf{E}(M_t^2)} \xrightarrow{\mathbf{P}} 1.$$

*Then, as $t \to \infty$,*
$$\frac{M_t}{\sqrt{\mathbf{E}(M_t^2)}} \xrightarrow{d} \mathcal{N}(0,1).$$

The idea for the proof of the main result below is inspired by the proof of Theorem 3.3 in [Dufresne, 2004].

**Theorem 4.4.4.** *Assume that $\hat{R}$ is a non-deterministic Lévy process with bounded jumps. Assume that $\mathbf{E}(\hat{R}_1) > 0$ and that*
$$\int_1^\infty \ln(x)|\Pi_X(dx)| < \infty.$$

*Additionally, assume that $\mathbf{P}((Y_t)^+ > 0) > 0$, $\mathbf{P}((Y_t)^- > 0) > 0$ and that*
$$\mathbf{P}(\tilde{Z}_\infty + y = 0) = 0 \ \text{ and } \ \mathbf{P}(\tilde{Z}_t + y = 0) = 0, \ \forall t \geq 0. \tag{4.12}$$

*We have*
$$\frac{\ln\left((Y_t)^+\right) - d_R t}{\sqrt{k_R t}} \xrightarrow{d} \mathcal{N}(0,1) \ \text{ and } \ \frac{\ln\left((Y_t)^-\right) - d_R t}{\sqrt{k_R t}} \xrightarrow{d} \mathcal{N}(0,1),$$

*where* $d_R = \mathbf{E}(\hat{R}_1) = a_R - \sigma_R^2/2$ *and*

$$k_R = \mathrm{Var}(\hat{R}_1) = \sigma_R^2 + \int_{\mathbb{R}} x^2 K_{\hat{R}}(dx) = \sigma_R^2 + \int_{-1}^{\infty} |\ln(1+x)|^2 K_R(dx).$$

*Proof.* Let $b > 0$ be a bound on the jumps of $\hat{R}$. By Proposition 1.2.12, the characteristics of $\hat{R}$ (for the characteristic function $h(x) = \mathbf{1}_{\{|x| \leq b\}}$), are $(d_R, \sigma_R^2, K_{\hat{R}})$ and the Lévy-Itô decomposition is

$$\hat{R}_t = d_R t + \sigma_R W_t^{(2)} + \int_{0+}^{t} \int_{\{|x| \leq b\}} x(\mu_{\hat{R}}(ds, dx) - K_{\hat{R}}(dx)ds).$$

Thus,

$$\frac{\hat{R}_t - d_R t}{\sqrt{k_R t}} = \frac{M_t}{\sqrt{k_R t}},$$

where $M = (M_t)_{t \geq 0}$ is given by

$$M_t = \sigma_R W_t^{(2)} + \int_{0+}^{t} \int_{\mathbb{R}} x(\mu_{\hat{R}}(ds, dx) - K_{\hat{R}}(dx)ds), \;\; t \geq 0.$$

Here, we replaced the set $\{|x| \leq b\}$ in the integral by $\mathbb{R}$ since the jumps are bounded by $b$.

We will start by proving that, as $t \to \infty$,

$$\frac{M_t}{\sqrt{k_R t}} \xrightarrow{d} \mathcal{N}(0, 1).$$

We separate into two cases (i) $\int_{\mathbb{R}} x^2 K_{\hat{R}}(dx) = 0$ and (ii) $\int_{\mathbb{R}} x^2 K_{\hat{R}}(dx) > 0$.

First, for case (i), note that the assumption implies

$$\mathbf{E}\left(\sum_{0<s\leq t} (\Delta \hat{R}_s)^2\right) = \mathbf{E}\left(\int_{0+}^{t} \int_{\mathbb{R}} x^2 K_{\hat{R}}(dx)ds\right) = 0,$$

and so $\Delta \hat{R}_t = 0$ ($\mathbf{P} - a.s.$), for all $t \geq 0$. Thus,

$$\frac{\hat{R}_t - d_R t}{\sqrt{k_R t}} = \frac{\sigma_R W_t^{(2)}}{\sigma_R \sqrt{t}} \xrightarrow{d} \mathcal{N}(0, 1).$$

For case (ii), we will use Lemma 4.4.3. Using the independence of the terms in the Lévy-Itô decomposition and Theorem 1 p.176 in [Liptser and Shiryayev, 1989], we obtain

$$\mathbf{E}(M_t^2) = \sigma_R^2 t + \mathbf{E}\left(\int_{0+}^t \int_{\mathbb{R}} x^2 K_{\hat{R}}(dx)ds\right) = k_R t.$$

Thus, $M$ is locally square integrable and the first assumption of Lemma 4.4.3 is satisfied.

The second assumption is also satisfied since, as $t \to \infty$,

$$\frac{\mathbf{E}(\sup_{0\le s\le t} |\Delta M_s|)}{\sqrt{\mathbf{E}(M_t^2)}} = \frac{\mathbf{E}(\sup_{0\le s\le t} |\Delta \hat{R}_s|)}{\sqrt{k_R t}} \le \frac{b}{\sqrt{k_R t}} \to 0.$$

For the third assumption, note that

$$[M]_t = \langle M^c\rangle_t + \sum_{0<s\le t} (\Delta M_t)^2 = \sigma_R^2 t + \int_{0+}^t \int_{\mathbb{R}} x^2 \mu_{\hat{R}}(ds, dx), \;\; t \ge 0.$$

But,

$$\frac{[M]_t}{\mathbf{E}(M_t^2)} = \frac{\sigma_R^2}{k_R} + \frac{1}{k_R} \frac{\int_{0+}^t \int_{\mathbb{R}} x^2 \mu_{\hat{R}}(ds, dx)}{\int_{0+}^t \int_{\mathbb{R}} x^2 K_{\hat{R}}(dx)ds} \left(\int_{\mathbb{R}} x^2 K_{\hat{R}}(dx)\right).$$

Now, Theorem 12 p.145 in [Liptser and Shiryayev, 1989] tells us that for a locally integrable increasing process $A$, we have $A_t/\tilde{A}_t \xrightarrow{\mathbf{P}} 1$, as $t \to \infty$, where $\tilde{A}$ is the compensator of $A$ when $\mathbf{P}(\tilde{A}_\infty = \infty) = 1$ and

$$\mathbf{E}\left(\sup_{t\ge 0} \Delta A_t\right) < \infty. \tag{4.13}$$

Thus, letting

$$A_t = \int_{0+}^t \int_{\mathbb{R}} x^2 \mu_{\hat{R}}(ds, dx) \;\text{ and }\; \tilde{A}_t = \int_{0+}^t \int_{\mathbb{R}} x^2 K_{\hat{R}}(dx)ds = t \int_{\mathbb{R}} x^2 K_{\hat{R}}(dx)$$

we see that indeed $A$ is locally integrable increasing and that $\mathbf{P}(\tilde{A}_\infty = \infty) = 1$ when $\int_{\mathbb{R}} x^2 K_{\hat{R}}(dx) > 0$. For (4.13), note that $\Delta A_t = (\Delta \hat{R}_t)^2 \le b^2$, for all $t \ge 0$, and thus

$$\mathbf{E}\left(\sup_{t\ge 0} \Delta A_t\right) \le b^2 < \infty.$$

From Lemma 4.4.3, we then obtain

$$\frac{\hat{R}_t - d_R t}{\sqrt{k_R t}} \xrightarrow{d} \mathcal{N}(0,1), \tag{4.14}$$

as $t \to \infty$.

Now, note that from $\mathbf{P}(\tilde{Z}_t + y = 0) = 0$, we obtain $\mathbf{P}(\tilde{Z}_t + y \neq 0) = 1$ and so, for all $t \geq 0$,

$$(Y_t)^+ = (Y_t)^+ \mathbf{1}_{\{\tilde{Z}_t + y \neq 0\}} = e^{\hat{R}_t}(y + \tilde{Z}_t)^+ \mathbf{1}_{\{\tilde{Z}_t + y \neq 0\}} \ \ (\mathbf{P} - a.s.)$$

or equivalently

$$\ln\left((Y_t)^+\right) = \hat{R}_t + \ln\left((y + \tilde{Z}_t)^+ \mathbf{1}_{\{\tilde{Z}_t + y \neq 0\}}\right) \ \ (\mathbf{P} - a.s.)$$

or

$$\frac{\ln\left((Y_t)^+\right) - d_R t}{\sqrt{k_R t}} = \frac{\hat{R}_t - d_R t}{\sqrt{k_R t}} + \frac{\ln\left((y + \tilde{Z}_t)^+ \mathbf{1}_{\{\tilde{Z}_t + y \neq 0\}}\right)}{\sqrt{k_R t}} \ \ (\mathbf{P} - a.s.).$$

But, Lemma 4.4.1 implies that $(\tilde{Z}_t)_{t \geq 0}$ converges to a finite random variable $(\mathbf{P} - a.s.)$, and so, as $t \to \infty$,

$$\ln\left((y + \tilde{Z}_t)^+ \mathbf{1}_{\{\tilde{Z}_t + y \neq 0\}}\right) \to \ln\left((y + \tilde{Z}_\infty)^+ \mathbf{1}_{\{\tilde{Z}_\infty + y \neq 0\}}\right) \ \ (\mathbf{P} - a.s.).$$

From Assumption (4.12), we then obtain $\mathbf{P}(\tilde{Z}_\infty + y \neq 0) = 1$ and so the limiting random variable is finite $(\mathbf{P} - a.s.)$. Thus, (4.14) and Slutsky's lemma yield the final result. The proof for $(Y_t)^-$ is obtained by replacing $(Y_t)^+$ by $(Y_t)^-$ in the above arguments. □

**Remark 4.4.5.** An adaptation of the proof of Theorem 2.2.3 shows that we have $\mathbf{P}(\tilde{Z}_t + y = 0) = 0$, for all $t \geq 0$, when $\sigma_X > 0$ or $K_X(\mathbb{R}) = \infty$. The condition $\mathbf{P}(\tilde{Z}_\infty + y = 0) = 0$ seems harder to verify. Some results concerning the absolute continuity of $\tilde{Z}_\infty$ are given in [Bertoin et al., 2008].

Thus, we can suggest large-time approximating distributions.

**Large-time Approximating Distribution.** When $\mathbf{E}(\hat{R}_1) < 0$ (large volatility) and under the assumptions of Lemma 4.4.1, we suggest to use the distribution of

$$Z_\infty = \int_{0+}^{\infty} e^{\hat{R}_{s-}} dX_s$$

as an approximating distribution for $Y_t$ when $t$ is large.

When $\mathbf{E}(\hat{R}_1) > 0$ (small volatility) and under the additional assumptions of Theorem 4.3.1, we suggest to use the log-normal distribution defined by the following density

$$g_{LN}(v) = \frac{1}{v\sqrt{2\pi k_R t}} \exp\left(-\frac{(\ln(v) - d_R t)^2}{2k_R t}\right)$$

where $d_R = \mathbf{E}(\hat{R}_1) = a_R - \sigma_R^2/2$ and

$$k_R = \mathrm{Var}(\hat{R}_1) = \sigma_R^2 + \int_{-1}^{\infty} |\ln(1+x)|^2 K_R(dx),$$

as an approximating distribution for the laws of $(Y_t)^+$ and $(Y_t)^-$ when $t$ is large.

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**Titre :** Les Processus d'Ornstein-Uhlenbeck Généralisés en Théorie de la Ruine

**Mots clés :** semimartingales, processus d'Ornstein-Uhlenbeck généralisés, processus autorégressifs, théorèmes limites, problème de la ruine avec investissement, lois à temps fixes.

**Résumé :** Cette thèse contribue à l'étude des processus d'Ornstein-Uhlenbeck généralisés (GOU) et de leurs applications en théorie de la ruine. Les processus GOU, qui sont les solutions de certaines équations différentielles stochastiques linéaires, ont été introduits en théorie de la ruine par Paulsen en 1993 en tant que modèles pour le capital d'une assurance soumise au risque de marché. En général, ces processus sont choisis comme modèles de manière a priori. La première et principale contribution de cette thèse est de montrer que les processus GOU apparaissent de manière naturelle comme limites faibles de processus autorégressifs à coefficients aléatoires qui sont très utilisés en probabilité appliquée. À partir de ce résultat, la convergence des temps de ruine, des probabilités de ruine et des moments est aussi démontrée.

Le reste de la thèse traite de certaines propriétés des processus GOU. En particulier, le problème de la ruine est traité et de nouvelles bornes sur les probabilités de ruine sont obtenues. Ces résultats généralisent aussi des résultats connus au cas où le risque de marché est modélisé par une semimartingale.

La dernière partie de la thèse s'éloigne de la théorie de la ruine pour passer à l'étude de la loi du processus à temps fixe. En particulier, une équation intégro-différentielle partielle pour la densité est obtenue, ainsi que des approximations pour la loi en temps courts et longs. Cet éloignement de la probabilité de ruine s'explique par le fait que la plupart des mesures de risques utilisées dans la pratique sont basées sur ces lois.

**Title :** Generalized Ornstein-Uhlenbeck Processes in Ruin Theory

**Keywords :** semimartingales, generalized Ornstein-Uhlenbeck processes, autoregressive processes, limit theorems, ruin problem with investment, laws at fixed times.

**Abstract :** This thesis is concerned with the study of Generalized Ornstein-Uhlenbeck (GOU) processes and their application in ruin theory. The GOU processes, which are the solutions of certain linear stochastic differential equations, have been introduced in ruin theory by Paulsen in 1993 as models for the surplus capital of insurance companies facing both insurance and market risks. In general, these processes were chosen as suitable models on an a priori basis.

The first and main contribution of this thesis is to show that GOU processes appear naturally as weak limits of random coefficient autoregressive processes which are used extensively in applied probability. Using this result, the convergence in distribution of the ruin times, the convergence of the ultimate ruin probability and the moments are also shown.

The rest of the thesis deals with the study of certain properties of GOU processes. In particular, the ruin problem for the GOU process is studied and new bounds on the ruin probabilities are obtained. These results also generalize some known upper bounds, asymptotic results and conditions for certain ruin to the case when the market risk is modelled by a semimartingale.

The final section of the thesis moves away from classical ruin theory and lays some first directions for the study of the law of GOU processes at fixed times. In particular, a partial integro-differential equation for the density, large and small-time asymptotics are obtained for these laws. This shift away from the ruin probability is explained by the fact that most risk measures used in practice such as Value-at-Risk are based on these laws instead.