Generalized Ornstein-Uhlenbeck Processes in Ruin Theory

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Goal and Contributions of the Thesis

Goal :

- motivate Generalized Ornstein-Uhlenbeck (GOU) processes as suitable models in ruin theory
- contribute to the study of some of their properties

1. show that GOU processes appear as weak limits of discrete-time processes
2. study ruin problem for GOU processes when the investment process is a general semimartingale
3. lay some directions for the study of the law at fixed time of GOU processes

What are Generalized Ornstein-Uhlenbeck (GOU) processes ?

The (classical) Ornstein-Uhlenbeck process

Ornstein-Uhlenbeck process ([Uhlenbeck and Ornstein, 1930]) : model for the motion of a particle in a fluid which is subjected a frictional force.

Stochastic differential equation :

$$dY_t = dB_t - \lambda Y_t dt, \quad t \geq 0,$$

- $Y_0 = y \in \mathbb{R}$: starting point of the process,
- $\lambda > 0$: parameter,
- and $B = (B_t)_{t \geq 0}$: standard Brownian motion.

Explicit expression :

$$Y_t = e^{-\lambda t} \left(y + \int_0^t e^{-\lambda s} dB_s \right), \quad t \geq 0.$$

Generalization : linear stochastic differential equation

$$dY_t = dX_t + Y_{t-}dR_t, \quad t \geq 0,$$

- Y_0 : $\mathcal{F}_0$-measurable random variable
- $X = (X_t)_{t \geq 0}$ and $R = (R_t)_{t \geq 0}$: two semimartingales

(i) $R_t = -\lambda t, X_t = B_t$ and $Y_0 = y \in \mathbb{R}$ $(\mathbf{P} - a.s.) \Longrightarrow$ (classical) Ornstein-Uhlenbeck process

(ii) $X_t = 0$ and $Y_0 = 1$ $(\mathbf{P} - a.s.) \Longrightarrow$ Doléans-Dade exponential, explicitly :

$$Y_t = \mathcal{E}(R)_t = \exp\left(R_t - \frac{1}{2}\langle R^c \rangle_t\right) \prod_{0<s\leq t} (1 + \Delta R_s)e^{-\Delta R_s}, \quad t \geq 0$$

Proposition

Assume that $\mathsf{P}(\Delta R_t > -1, \forall t \geq 0) = 1$, *that* X *and* R *are independent semimartingales and that either* X *or* R *is a Lévy process. Then, the unique solution of the equation (with* $Y_0 = y > 0$ $(\mathsf{P} - a.s.)$*) can be written explicitly as*

$$Y_t = \mathcal{E}(R)_t \left(y + \int_{0+}^{t} \mathcal{E}(R)_{s-}^{-1} dX_s \right) = e^{\hat{R}_t} \left(y + \int_{0+}^{t} e^{-\hat{R}_{s-}} dX_s \right),$$

where $\hat{R}_t = \ln(\mathcal{E}(R)_t)$*, for all* $t \geq 0$*. We call this process the generalized Ornstein-Uhlenbeck (GOU) process associated to* X *and* R*.*

What is Ruin Theory ?

The object of Ruin Theory

Ruin Theory : study of the possibility of insolvency in insurance practice.

Main questions :

1. What is a good model ? What kind of risks to consider ?
2. Given a model, what is the probability of ruin or insolvency ?

Ruin Times and Ruin Probabilities

$S = (S_t)_{t \geq 0}$: continuous-time stochastic process with $S_0 = x > 0$ ($\mathsf{P} - a.s.$)

The ruin time of S is :

$$\tau(x) = \inf\{t > 0 : S_t < 0\}$$

with $\inf\{\emptyset\} = \infty$.

- (finite-time) ruin probability : $\mathsf{P}(\tau(x) \leq T)$, for $T \geq 0$
- ultimate ruin probability : $\mathsf{P}(\tau(x) < \infty)$

Note : Dependence on the "initial capital" x.

A classic in Ruin Theory : the Cramér-Lundberg model

Classic model :

$$X_t = x + pt - \sum_{i=1}^{N_t} Z_i, \quad t \geq 0,$$

- $x > 0$: the initial capital
- $p > 0$: mean income received from premiums
- $N = (N_t)_{t\geq 0}$: a standard Poisson process representing the random number and times of payments
- and $(Z_i)_{i\in\mathbb{N}^*}$: a sequence of non-negative random variables representing the sizes of the payments

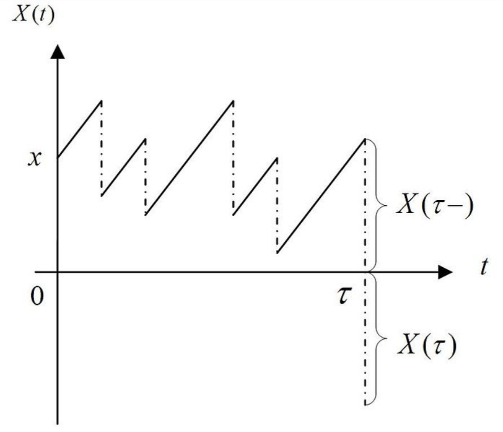

Figure 1: The ruin time for the Cramér-Lundberg model $X = (X_t)_{t \geq 0}$ starting in $X_0 = x$ ($\mathrm{P} - a.s.$). *picture by R. Feng (CC BY-SA 3.0)*

In the Cramér-Lundberg model : only one risk due to underwriting insurance contracts.

In practice : insurance risk plus market risk.

So we have two processes

- $X = (X_t)_{t\geq 0}$: underwriting or insurance risk
- $(R_t)_{t\geq 0}$: market risk

[Paulsen, 1993] suggests X and R semimartingales and

$$dY_t = dX_t + Y_{t-}dR_t, \quad t \geq 0,$$

with $Y_0 = y > 0$ $(\mathrm{P} - a.s.)$.

1. GOU Processes as Weak Limits

First and main contribution of the thesis : the GOU process as weak limit of random coefficient autoregressive process of order 1 $\theta_0 = y$ and

$$\theta_k = \xi_k + \theta_{k-1}\rho_k, \quad k \in \mathbb{N}^*.$$

$(\xi_k)_{k\in\mathbb{N}^*}$ and $(\rho_k)_{k\in\mathbb{N}^*}$: two independent and i.i.d. sequences of random variables with $\rho_k > 0$ ($\mathsf{P} - a.s.$), for all $k \in \mathbb{N}^*$.

Actuarial approximations

[Iglehart, 1969] : weak convergence of a sequence of Cramér-Lundberg processes and approximation of the ruin probability.

[Paulsen and Gjessing, 1997] : weak convergence of a sequence of GOU processes and approximation of the ruin probabilities.

Weak convergence of discrete-time processes

[Cumberland and Sykes, 1982] : weak convergence to a standard OU process when ρ_k deterministic and constant.

[Dufresne, 1989] : weak convergence to a modified OU process when ξ_k deterministic and satisfying some regularity conditions.

[Duffie and Protter, 1992] : weak convergence to the Doléans-Dade exponential when $\xi_k = 0$, for all $k \in \mathbb{N}^*$.

Main Assumption

Assumption (H^2)

We say that a random variable Z satisfies (H^2) *if Z is square-integrable with $\mathrm{Var}(Z) > 0$, where $\mathrm{Var}(Z)$ is the variance of Z.*

Remark : In the thesis, a slightly weaker assumption is used.

Assumption : ξ_1 and $\ln(\rho_1)$ both satisfy (H^2)

Divide the time interval into $n \in \mathbb{N}^*$ subintervals of length $1/n$ and let

$$\theta^{(n)}\left(\frac{k}{n}\right) = \xi_k^{(n)} + \theta^{(n)}\left(\frac{k-1}{n}\right)\rho_k^{(n)}, k \in \mathbb{N}^*.$$

To define $(\xi_k^{(n)})_{k\in\mathbb{N}^*}$ and $(\xi_k^{(n)})_{k\in\mathbb{N}^*}$ follow **[Dufresne, 1989]** :

$$\xi_k^{(n)} = \frac{\mu_\xi}{n} + \frac{\xi_k - \mu_\xi}{\sqrt{n}}, \quad \mu_\xi = \mathrm{E}(\xi_1)$$

and $\rho_k^{(n)} = \exp(\gamma_k^{(n)})$

$$\gamma_k^{(n)} = \frac{\mu_\rho}{n} + \frac{\ln(\rho_k) - \mu_\rho}{\sqrt{n}}, \quad \mu_\rho = \mathrm{E}(\ln(\rho_1)).$$

Continuous-time : $\theta^{(n)} = (\theta_t^{(n)})_{t \geq 0}$ as $\theta_t^{(n)} = \theta^{(n)}([nt]/n)$, where [.] is the floor function.

Theorem (Invariance principle)

Assume that ξ_1 and $\ln(\rho_1)$ both satisfy (H^2), then $\theta^{(n)} \xrightarrow{d} Y$, as $n \to \infty$, for $Y = (Y_t)_{t \geq 0}$ defined by

$$Y_t = e^{\hat{R}_t} \left(y + \int_{0+}^{t} e^{-\hat{R}_{s-}} dX_s \right), t \geq 0,$$

with $\hat{R}_t = \mu_\rho t + \sigma_\rho W_t$ and $X_t = \mu_\xi t + \sigma_\xi \tilde{W}_t$, for all $t \geq 0$, where $(W_t)_{t \geq 0}$ and $(\tilde{W}_t)_{t \geq 0}$ are two independent standard Brownian motions and $\sigma_\xi^2 = \mathrm{Var}(\xi_1)$ and $\sigma_\rho^2 = \mathrm{Var}(\ln(\rho_1))$.

Define, for $n \geq 1$,

$$\tau^n(y) = \inf\{t > 0 : \theta_t^{(n)} < 0\}$$

and also

$$\tau(y) = \inf\{t > 0 : Y_t < 0\}.$$

Theorem

Assume that ξ_1 and $\ln(\rho_1)$ both satisfy (H^2). We have, for all $T \geq 0$,

$$\lim_{n \to \infty} \mathrm{P}(\tau^n(y) \leq T) = \mathrm{P}(\tau(y) \leq T)$$

and, equivalently, $\tau^n(y) \xrightarrow{d} \tau(y)$, as $n \to \infty$.

Invariance principle : theoretical justification for the GOU process (in the spirit of [Duffie and Protter, 1992]).

Convergence : approximations for the values of certain functionals.

Turning to the Ultimate Ruin Probability

We want convergence for the ultimate ruin probability $\mathsf{P}(\tau(y) < \infty)$.

Assume that (H^2) holds. When $\mu_\rho \leq 0$ (or equivalently $\mathsf{P}(\tau(y) < \infty) = 1$, for all $y > 0$), we have

$$\lim_{n \to \infty} \mathsf{P}(\tau^n(y) < \infty) = 1.$$

What happens when the limiting probability is not 1 ?

Theorem

Assume that ξ_1 *and* $\ln(\rho_1)$ *both satisfy* (H^2). *When* $\mu_\rho > 0$, *we assume additionally that there exists* $C < 1$ *and* $n_0 \in \mathbb{N}^*$ *such that*

$$\sup_{n \geq n_0} \mathrm{E}\left(e^{-2\gamma_1^{(n)}}\right)^n = \sup_{n \geq n_0} \mathrm{E}\left((\rho_1^{(n)})^{-2}\right)^n \leq C.$$

Then,

$$\lim_{n \to \infty} \mathrm{P}(\tau^n(y) < \infty) = \mathrm{P}(\tau(y) < \infty) = \frac{H(-y)}{H(0)}$$

where $H : \mathbb{R}_+ \to \mathbb{R}_+^*$ *is given in* [Paulsen and Gjessing, 1997].

Example (Normal log-returns)

Take ξ_1 to be any random variable satisfying (H^2) *and* $\ln(\rho_1) \sim \mathcal{N}(\mu_\rho, \sigma_\rho^2)$, *with* $\mu_\rho > 0$, *then*

$$\mathsf{E}\left(e^{-2\gamma_1^{(n)}}\right)^n = e^{-2(\mu_\rho - \sigma_\rho^2)},$$

for all $n \in \mathbb{N}^*$, *so* $n_0 = 1$ *and the condition* $C < 1$ *is equivalent to* $\mu_\rho > \sigma_\rho^2$.

We know that $\theta_1^{(n)} \xrightarrow{d} Y_1$.

Natural question : do the moments of $\theta_1^{(n)}$ converge to the moments of Y_1.

Theorem

Assume that ξ_1 *and* $\ln(\rho_1)$ *both satisfy* (H^2)*. Assume that* $\mathrm{E}(|\xi_1|^q) < \infty$*, and that*

$$\sup_{n\in\mathbb{N}^*} \mathrm{E}\left(e^{q\gamma_1^{(n)}}\right)^n = \sup_{n\in\mathbb{N}^*} \mathrm{E}\left((\rho_1^{(n)})^q\right)^n < \infty,$$

for some integer $q \geq 2$*. Then, for each* $p \in \mathbb{N}^*$ *such that* $1 \leq p < q$*, we have*

$$\lim_{n\to\infty} \mathrm{E}[(\theta_1^{(n)})^p] = \mathrm{E}[(Y_1)^p] = m_p(1),$$

for a function m_p *that can be computed recursively.*

2. On the Ruin Problem for GOU Processes

Assumptions and Goals

$X = (X_t)_{t \geq 0}$ and $R = (R_t)_{t \geq 0}$: two independent stochastic processes

- X : Lévy process with characteristics (a_X, σ_X^2, K_X)
- R : a general semimartingale with $\mathrm{P}(\Delta R_t > -1, \forall t \geq 0) = 1$

GOU process :

$$Y_t = \mathcal{E}(R)_t \left(y + \int_{0+}^{t} \mathcal{E}(R)_{s-}^{-1} dX_s \right), \quad t \geq 0.$$

Goal : obtain results about $\mathrm{P}(\tau(y) \leq T)$, for $T > 0$, and $\mathrm{P}(\tau(y) < \infty)$ for the GOU process.

Related Literature

Well studied problem but mainly when R is also a Lévy process.

[Frolova et al., 2002], [Kalashnikov and Norberg, 2002] and [Paulsen, 2002], [Paulsen, 1998] : there exists $\beta \in \mathbb{R}$ (depending only on R) such that

- (small volatility) when $\beta > 0$, $\mathrm{P}(\tau(y) < \infty)$ behaves essentially like $y^{-\beta}$
- (large volatility) when $\beta \leq 0$, $\mathrm{P}(\tau(y) < \infty) = 1$, for all $y > 0$

When R is a semimartingale

[Hult and Lindskog, 2011] : asymptotic results as $y \to \infty$, for the finite-time ruin probability when X has a regularly varying left-tail.

The Contributions of the Thesis

Focus for ruin theory : on the investment strategy which can be modelled by a semimartingale.

First contribution : extend some results which where known in the Lévy case.

Second contribution : shift concern from $P(\tau(y) < \infty)$ to $P(\tau(y) \leq T)$, for some $T > 0$, and away from asymptotic results to inequalities which hold for every $y > 0$.

Exponential Functionals

The exponential functionals associated to R :

$$J_T(\alpha) = \int_0^T e^{-\alpha \hat{R}_s} ds, \quad I_T = J_T(1) \text{ and } J_T = J_T(2)$$

where $\alpha > 0$, $T \in (0, \infty]$ and $\hat{R}_t = \ln \mathcal{E}(R)_t$, for all $t \geq 0$.

Define

$$\beta_T = \sup \left\{ \beta \geq 0 : \mathrm{E}(J_T^{\beta/2}) < \infty, \mathrm{E}(J_T(\beta)) < \infty \right\}$$

and

$$\beta_\infty = \sup \left\{ \beta \geq 0 : \mathrm{E}(I_\infty^\beta) < \infty, \mathrm{E}(J_\infty^{\beta/2}) < \infty, \mathrm{E}(J_\infty(\beta)) < \infty \right\}.$$

Upper bound on the Ruin Probability

Theorem

Let $T > 0$. Assume that $\beta_T > 0$ and that, for some $0 < \alpha < \beta_T$, we have $\mathbf{E}(|X_1|^\alpha) < \infty$. Then, for all $y > 0$,

$$\mathsf{P}(\tau(y) \leq T) \leq \frac{C_1\mathbf{E}(I_T^\alpha) + C_2\mathbf{E}(J_T^{\alpha/2}) + C_3\mathbf{E}(J_T(\alpha))}{y^\alpha},$$

where the expectations on the right hand side are finite and $C_1 \geq 0$, $C_2 \geq 0$, and $C_3 \geq 0$ are constants. Moreover, if $\mathbf{E}(|X_1|^\alpha) < \infty$, for all $0 < \alpha < \beta_T$, then

$$\limsup_{y\to\infty} \frac{\ln\left(\mathsf{P}(\tau(y) \leq T)\right)}{\ln(y)} \leq -\beta_T.$$

Letting $T \to \infty$ and using the monotone convergence theorem we get.

Corollary

Assume that $\beta_\infty > 0$ *and that* $\mathsf{E}(|X_1|^\alpha) < \infty$, *for some* $0 < \alpha < \beta_\infty$, *then*

$$\mathsf{P}(\tau(y) < \infty) \leq \frac{C_1 \mathsf{E}(I_\infty^\alpha) + C_2 \mathsf{E}(J_\infty^{\alpha/2}) + C_3 \mathsf{E}(J_\infty(\alpha))}{y^\alpha},$$

where $C_1 \geq 0$, $C_2 \geq 0$, *and* $C_3 \geq 0$ *are constants. Moreover, if* $\mathsf{E}(|X_1|^\alpha) < \infty$, *for all* $0 < \alpha < \beta_\infty$, *then*

$$\limsup_{y \to \infty} \frac{\ln\left(\mathsf{P}(\tau(y) < \infty)\right)}{\ln(y)} \leq -\beta_\infty.$$

Are the bounds optimal in some sense ?

Theorem

Let $T > 0$. Assume that for $\gamma_T \geq 1$ we have $\mathsf{E}(I_T^{\gamma_T}) = \infty$. Additionally, assume that $\mathsf{E}(|X_1|) < \infty$ and that $\mathsf{E}(X_1) < 0$ or $\sigma_X > 0$. Then,

$$\limsup_{y\to\infty} \frac{\ln\left(\mathsf{P}(\tau(y) \leq T)\right)}{\ln(y)} \geq -\gamma_T.$$

Remark : When R is a Lévy process, we have $\mathsf{E}(I_T^{\beta_T}) = \infty$ and so

$$\limsup_{y\to\infty} \frac{\ln\left(\mathsf{P}(\tau(y) \leq T)\right)}{\ln(y)} = -\beta_T.$$

Computing the Values of β_T and β_∞

Problem : can β_T and β_∞ be computed and how are they related with the β appearing in the known results ?

When R is also a Lévy process, we get a simple method which, for ultimate ruin probability, coincides with the known results.

The study of β_T or β_∞ for more general processes R remains open.

Explicit Expression for β_∞

Proposition

Suppose that R is a Lévy process and that $\hat{R}$ admits a Laplace transform, for all $t \geq 0$, i.e. for $\alpha > 0$

$$\mathrm{E}(\exp(-\alpha \hat{R}_t)) = \exp(t\psi_{\hat{R}}(\alpha))$$

and that its Laplace exponent $\psi_{\hat{R}}$ has a strictly positive root β_0. Then, $\beta_\infty = \beta_0$.

Remark : The importance of the root of the Laplace exponent was already identified in **[Paulsen, 2002]**.

Computation of β_∞ in a Black-Scholes Market

Example

Suppose that $R_t = a_R t + \sigma_R W_t$, *for all* $t \geq 0$, *where* $a_R \in \mathbb{R}$, $\sigma_R > 0$ *and* $W = (W_t)_{t \geq 0}$ *is a standard Brownian motion, then* $\hat{R}_t = \left(a_R - \frac{\sigma_R^2}{2}\right) t + \sigma_R W_t$, *for all* $t \geq 0$.

Thus, we obtain $\psi_{\hat{R}}(\alpha) = -\left(a_R - \frac{1}{2}\sigma_R^2\right)\alpha + \frac{\sigma_R^2}{2}\alpha^2$ *and, by the above proposition,* $\beta_\infty = \frac{2a_R}{\sigma_R^2} - 1$. *This coincides with the known results in [Frolova et al., 2002] and [Kabanov and Pergamentshchikov, 2016].*

Conditions for Ruin with Probability One

Large volatility case in the semimartingale setting : we want to find conditions under which $P(\tau(y) < \infty) = 1$, for all $y > 0$.

Condition for Ruin with Probability One

Assumption : X has positive jumps bounded by $a > 0$ and satisfies

$$a_X < 0 \text{ or } \sigma_X > 0 \text{ or } K_X([-a, a]) > 0.$$

Theorem

In addition assume that $(\mathsf{P} - a.s.)$, $I_\infty = \infty$, $J_\infty = \infty$ *and that there exists a limit*

$$\lim_{t \to \infty} \frac{I_t}{\sqrt{J_t}} = L$$

with $0 < L < \infty$. *Then,* $\mathsf{P}(\tau(y) < \infty) = 1$, *for all* $y > 0$.

Corollary

Suppose that X satisfies the on of the conditions above the previous theorem. Moreover, suppose that R is a Lévy process with characteristic triplet (a_R, σ_R^2, K_R) *satisfying*

$$\int_{-1}^{\infty} |\ln(1+x)| \mathbf{1}_{\{|\ln(1+x)|>1\}} K_R(dx) < \infty$$

and

$$a_R - \frac{\sigma_R^2}{2} + \int_{-1}^{\infty} (\ln(1+x) - x\mathbf{1}_{\{|\ln(1+x)|\leq 1\}}) K_R(dx) < 0.$$

Then, $\mathrm{P}(\tau(y) < \infty) = 1$, *for all* $y > 0$.

3. On the Law at Fixed Time of GOU Processes

Assumptions and Goals

Assumption : X and R are two independent Lévy processes with characteristics (a_X, σ_X^2, K_X) and (a_R, σ_R^2, K_R) and $\mathsf{P}(\Delta R_t > -1, \forall t \geq 0) = 1$.

GOU process :

$$Y_t = \mathcal{E}(R)_t \left(y + \int_{0+}^{t} \mathcal{E}(R)_{s-}^{-1} dX_s \right), \quad t \geq 0.$$

In practice : risk measures based on quantiles of the distribution of Y_t rather than ruin probabilities.

Goal : give some directions for the study of the law of Y_t.

The law at fixed time of Y

[Brokate et al., 2008] : PDE for $f(t,x) = \mathrm{P}(Y_t > x)$ when X is the Cramér-Lundberg model and $y = 0$.

Existence and the identification of the stationary law of Y

This law can be viewed as

$$Z_\infty = \int_{0+}^{\infty} \mathcal{E}(R)_{s-} dX_s.$$

[Paulsen, 1993], **[Carmona, 1996]**, **[Behme and Lindner, 2015]** : PIDE for distribution, characteristic function and density of Z_∞.

[Gjessing and Paulsen, 1997] : list of explicit distributions for Z_∞ for different choices of X and R.

GOU Processes are Feller Processes

Feller processes : time-homogeneous Markov process for which the generator satisfies some additional regularity conditions.

[Behme and Lindner, 2015] : the GOU process is a Feller process and explicit expression for the generator.

Our first result is a simple application of this fact and standard results about Feller processes.

Theorem

Assume that $K_R((-1,\infty)) < \infty$*, that* $K_X(\mathbb{R}) < \infty$ *and that* Y_t *admits a density* $p \in \mathcal{C}^{1,2}((0,T)\times\mathbb{R})$*, for all* $t \in (0,T)$*. Then,* $\partial_t p(t,x) =$

$$\frac{\sigma_X^2}{2}\partial_{xx}p(t,x) + \frac{\sigma_R^2}{2}\partial_{xx}\left(x^2 p(t,x)\right) - a_X\partial_x p(t,x) - a_R\partial_x(xp(t,x))$$
$$+ \int_{-1}^{\infty}\left(\frac{p(t,x(1+z)^{-1})}{1+z} - p(t,x) + z\mathbf{1}_{\{|z|\le 1\}}\partial_x(xp(t,x))\right)K_R(dz)$$
$$+ \int_{\mathbb{R}}\left(p(t,x-z) - p(t,x) + z\mathbf{1}_{\{|z|\le 1\}}\partial_x p(t,x)\right)K_X(dz)$$

for all $(t,x) \in (0,T)\times\mathbb{R}$*, with initial condition* $p(0,x) = \delta_y(x)$*, where* δ_y *is the Dirac measure at* $y > 0$*.*

Notation : $p^y(t,x)$ instead of $p(t,x)$, since p depends on the initial value $y > 0$.

Proposition

Assume that

1. $\mathbf{E}(|X_1|^p) < \infty$ *and* $\mathbf{E}(|R_1|^p) < \infty$, *for all* $p \in \mathbb{N}^*$,
2. $\sigma_X > 0$,
3. *there exists* $\rho > 0$ *such that* $K_R((-\infty, -1+\rho]) = 0$.

Then, for all $T > 0$, *the function* $(y,t,x) \mapsto p^y(t,x)$ *is of class* $\mathcal{C}^\infty(\mathbb{R}_+^* \times (0,T) \times \mathbb{R})$.

Problems : restrictive assumptions for the existence of p and equation hard to integrate without some numerical method.

$\Longrightarrow$ Focus on approximations for the laws of Y_t, when t is either small or large.

Exponential functional of R :

$$I_t = \int_0^t e^{\hat{R}_{s-}} ds, \quad t \geq 0,$$

where $\hat{R}_t = \ln \mathcal{E}(R)_t$, for all $t \geq 0$.

Theorem

Assume that $R_t = a_R t + \sigma_R W_t$, *for all* $t \geq 0$, *with* $\sigma_R > 0$ *and where* $W = (W_t)_{t \geq 0}$ *is a Brownian motion with drift.*
Additionally assume that

$$\int_{\mathbb{R}} |x| K_X(dx) < \infty.$$

Then, as $t \to 0+$,

$$\frac{Y_t - y - \delta_X I_t}{\sqrt{\sigma(y) I_t}} \xrightarrow{d} \mathcal{N}(0, 1),$$

where $\sigma(y) = \sigma_X^2 + \sigma_R^2 y^2$ *and* $\delta_X = \mathsf{E}(X_1)$.

[Dufresne, 2004] : as $t \to 0+$,

$$\frac{\ln(I_t) - \ln(t)}{\sigma_R \sqrt{t/3}} \xrightarrow{d} \mathcal{N}(0,1).$$

$\Longrightarrow$ approximation of the law of Y_t when t is small by a normal-log-normal mean variance mixture :

$$f_{NLN}(u,t) = \int_0^\infty \frac{1}{\sqrt{2\pi\sigma(y)v}} \exp\left(-\frac{(u-y-\delta_X v)^2}{2\sigma(y)v}\right) g_{LN}(v,t)dv,$$

where $\sigma(y) = \sigma_X^2 + \sigma_R^2 y^2$ and

$$g_{LN}(v,t) = \frac{1}{\sigma_R v}\sqrt{\frac{3}{2\pi t}} \exp\left(-\frac{3(\ln(v) - \ln(t))^2}{2\sigma_R^2 t}\right).$$

Define

$$Z_t = \int_{0+}^{t} e^{\hat{R}_{s-}} dX_s \text{ and } \tilde{Z}_t = \int_{0+}^{t} e^{-\hat{R}_{s-}} dX_s$$

and $\Pi_X(x) = K_X((-\infty, -x]) + K_X([x, \infty))$.

Lemma

Assume that $\mathsf{E}(|\hat{R}_1|) < \infty$ *and* $\int_1^\infty \ln(x)|\Pi_X(dx)| < \infty$.

1. *(Large volatility case) If* $\mathsf{E}(\hat{R}_1) < 0$, *then* $(Z_t)_{t\geq 0}$ *converges to finite random variables* Z_∞ $(\mathsf{P} - a.s.)$ *and* $Y_t \xrightarrow{d} Z_\infty$, *as* $t \to \infty$.
2. *(Small volatility case) If* $\mathsf{E}(\hat{R}_1) > 0$, *then* $(\tilde{Z}_t)_{t\geq 0}$ *converges to a finite random variables* $\tilde{Z}_\infty$ $(\mathsf{P} - a.s.)$.

Theorem

Assume that $\hat{R}$ is a non-deterministic Lévy process with bounded jumps. Assume that $\mathsf{E}(\hat{R}_1) > 0$ *and that*

$$\int_1^\infty \ln(x)|\Pi_X(dx)| < \infty.$$

Additionally, assume that $\mathsf{P}(\tilde{Z}_\infty + y = 0) = 0$ *and* $\mathsf{P}(\tilde{Z}_t + y = 0) = 0$*, for all* $t \geq 0$*. Then,*

$$\frac{\ln\left((Y_t)^+\right) - d_R t}{\sqrt{k_R t}} \xrightarrow{d} \mathcal{N}(0,1) \ \text{ and } \ \frac{\ln\left((Y_t)^-\right) - d_R t}{\sqrt{k_R t}} \xrightarrow{d} \mathcal{N}(0,1),$$

where $d_R = \mathsf{E}(\hat{R}_1) = a_R - \sigma_R^2/2$ *and* $k_R = \mathrm{Var}(\hat{R}_1)$.

Conclusion

Recapitulation and Future

Combination of different results about GOU processes within the context of ruin theory.

GOU processes are quite simple **BUT**

- mathematical aspects can be complicated and interesting
- important insight into the problem of investment by insurance companies

GOU process are a good first step to large-scale modelling of insurance companies in view of better regulation and risk control.

Thank you!

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